18.022: Multivariable calculus — The change of variables theorem

The mathematical term for a change of variables is the notion of a diffeomorphism. A map $\mathbf{F}\colon U \to V$ between *open* subsets of $\mathbb{R}^n$ is a *diffeomorphism* if $\mathbf{F}$ is one-to-one and onto and both $\mathbf{F}\colon U \to V$ and $\mathbf{F}^{-1}\colon V \to U$ are differentiable. Since

$$\mathbf{F}^{-1}(\mathbf{F}(\mathbf{x})) = \mathbf{x}$$
$$\mathbf{F}(\mathbf{F}^{-1}(\mathbf{y})) = \mathbf{y}$$

and since both $\mathbf{F}$ and $\mathbf{F}^{-1}$ are differentiable, the chain rule shows

$$D(\mathbf{F}^{-1})(\mathbf{y}) \cdot D\mathbf{F}(\mathbf{x}) = I$$
$$D\mathbf{F}(\mathbf{x}) \cdot D(\mathbf{F}^{-1})(\mathbf{y}) = I.$$

Hence, if $\mathbf{F}\colon U \to V$ is a diffeomorphism, the matrix of partial derivatives $D\mathbf{F}(\mathbf{x})$ is invertible, for all $\mathbf{x} \in U$. The converse is almost true:

Theorem. *Let $U \subset \mathbb{R}^n$ be open, and let $\mathbf{F}\colon U \to \mathbb{R}^n$ be a function of class C^1. Assume that $\mathbf{F}$ is one-to-one and that, for all $\mathbf{x} \in U$, the derivative $D\mathbf{F}(\mathbf{x})$ is invertible. Then $V = \mathbf{F}(U) \subset \mathbb{R}^n$ is open and $\mathbf{F}\colon U \to V$ a diffeomorphism.*

We note that the matrix $D\mathbf{F}(\mathbf{x})$ is invertible if and only if the determinant $\det D\mathbf{F}(\mathbf{x})$ is non-zero. This determinant is called the *Jacobian* of $\mathbf{F}$ at $\mathbf{x}$. The change-of-variables theorem for double integrals is the following statement.

Theorem. *Let $\mathbf{F}\colon U \to V$ be a diffeomorphism between open subsets of $\mathbb{R}^2$, let $D^* \subset U$ and $D = \mathbf{F}(D^*) \subset V$ be bounded subsets, and let $f\colon D \to \mathbb{R}$ be a bounded function. Then*

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(\mathbf{F}(u,v))|\det D\mathbf{F}(u,v)|dudv.$$

The equality means that the left-hand integral exists if and only if the right-hand integral does and that, if so, the two integrals are equal. We also write

$$\frac{\partial(x,y)}{\partial(u,v)}(u,v) = \det D\mathbf{F}(u,v)$$

for the Jacobian.

Example. We wish to calculate the area of the region D of the plane bounded by the four curves $y = x^3$, $y = 2x^3$, $xy = 1$, and $xy = 3$. If we let $u = x^3/y$ and $v = xy$, then these bounds become $u = 1$, $u = 1/2$, $v = 1$, and $v = 3$, respectively. We show that this change of coordinates is a diffeomorphism. First, if both $u > 0$ and $v > 0$, we can solve for x and y in terms of u and v, $x = (uv)^{1/4} = u^{1/4}v^{1/4}$ and $y = v/x = u^{-1/4}v^{3/4}$. So the map

$$\mathbf{F}\colon (0,\infty) \times (0,\infty) \to (0,\infty) \times (0,\infty)$$

defined by the $\mathbf{F}(u,v) = (u^{1/4}v^{1/4}, u^{-1/4}v^{3/4})$ is a bijection whose inverse is the map given by $\mathbf{F}^{-1}(x,y) = (x^3/y, xy)$. Second, both $\mathbf{F}$ and $\mathbf{F}^{-1}$ are differentiable. Hence, $\mathbf{F}$ is a diffeomorphism.

We calculate the Jacobian. We have

$$\det D(\mathbf{F}^{-1})(x,y) = \det \begin{bmatrix} 3x^2/y & -x^3/y^2 \\ y & x \end{bmatrix} = 4x^3/y = 4u,$$

and hence,

$$\det D\mathbf{F}(u,v) = (\det D(\mathbf{F}^{-1})(x,y))^{-1} = \frac{1}{4u}.$$

We can now use the change-of-variables theorem to evaluate the area of D.

$$\begin{aligned}\text{area}(D) &= \iint_D dxdy = \iint_{D^*} |\det D\mathbf{F}(u,v)|dudv \\ &= \int_1^3 \int_{\frac{1}{2}}^1 \frac{1}{4u} dudv = \frac{\ln 2}{4}\int_1^3 dv = \frac{\ln 2}{2}.\end{aligned}$$

The change-of-variables theorem for triple integrals is entirely similar:

THEOREM. *Let $\mathbf{F}\colon U \to V$ be a diffeomorphism between open subsets of $\mathbb{R}^3$, let $W^* \subset U$ and $W = \mathbf{F}(W^*) \subset V$ be bounded subsets, and let $f\colon W \to \mathbb{R}$ be a bounded function. Then*

$$\iiint_W f(x,y,z)dxdydz = \iiint_{W^*} f(\mathbf{F}(u,v,w))|\det D\mathbf{F}(u,v,w)|dudvdw.$$