My research belongs into the general realm of low dimensional topology and symplectic geometry. Important progress in these areas started with the work of Donaldson [D1] on smooth four-manifolds, based on Yang-Mills instantons, and the work of Gromov [G] on symplectic manifolds, based on pseudoholomorphic curves. In both subjects Floer [F1, F2] then introduced a new approach to infinite dimensional Morse theory based on the respective moduli spaces of solutions to nonlinear partial differential equations. This lead the way towards the construction of various algebraic structures from moduli spaces of PDEs which capture topological properties of the underlying manifolds. My work revolves around the construction and relation of such structures. One of my large scale projects, described in Section 1, is the enlargement the Fukaya category, which captures all Floer theories on a given symplectic manifold, to a symplectic 2-category, which also encodes relations between different symplectic manifolds. For that purpose I introduced pseudoholomorphic quilts – a new type of PDE that couples pseudoholomorphic curves ("patches") in different symplectic manifolds via "seam conditions" relating their boundaries. This framework allows to relate symplectic geometry to different geometric categories as e.g. in applications to Mirror Symmetry (relating symplectic and algebraic geometry) and Topological Quantum Field Theories (a type of topological invariants that is compatible with cobordisms). The latter are outlined in Section 2 and have led me to develop "2 + 1 + 1 Floer field theory" – a blueprint for constructing topological invariants of 3- and 4-dimensional manifolds via decomposition of the manifolds and representation of the "generating pieces" in the symplectic category. The analytic motivation for the construction of topological invariants via the symplectic category is an adiabatic deformation of the instanton equation to the Cauchy-Riemann equation in symplectic moduli space of dimensionally reduced instantons. One case of this Atiyah-Floer conjecture was proven by Dostoglou-Salamon [DS], relating instanton invariants on cylindrical 3-manifolds with pseudoholomorphic Floer theory. Section 3 describes my work towards more general relations between instanton invariants of 3- and 4-manifolds and pseudoholomorphic quilt invariants via degenerations of 3- and 4-manifolds along decompositions arising from singular fibrations over $S^1$ resp. $S^2$, in which the singular locus gives rise to boundary or seam conditions. The bubbling analysis for adiabatic limits of instantons led me to study holomorphic curves in infinite dimensional spaces of connections. This experience then informed a (still speculative) approach towards studying non-squeezing effects in Hamiltonian PDE flows from an infinite dimensional symplectic point of view described in Section 4.

A more foundational step in the construction of algebraic invariants from moduli spaces of PDEs is the regularization of the usually highly singular moduli spaces that is needed to make counts or integration well defined. For a small but highly significant class of moduli problems, regularization can be achieved by a geometric perturbation of the PDE. However, most spaces of pseudoholomorphic curves require an abstract regularization scheme for moduli spaces of PDEs. This can be separated into an analytic challenge – finding a compactification of the moduli space and describing it as the zero set of some type of section over some type of space – and the differential-topological challenge of associating to this abstract type of section a well defined count, fundamental class, or cobordism class of smooth spaces. Despite a number of promising approaches, the intricacy of these challenges and a lack of clear expositions has been hampering the entry of young researchers and sustainable progress in the field. I have thus embarked on a more systematic study of regularization techniques, of which Section 5 gives a somewhat detailed exposition to clarify the math-cultural dimension of this work.

Finally, Sections 5.4–5.7 outline the various resulting infrastructure projects from revisiting classical Morse theory and Gromov-Witten moduli spaces via the development of polyfold descriptions for Lagrangian Floer theories, the Fukaya category, and general quilt invariants to a new proof of the classical Arnold conjecture, which bounds the number of periodic orbits of a Hamiltonian system in terms of the total rank of homology of the symplectic phase space.
1. Pseudoholomorphic Quilts and the Symplectic Category

In work with Chris Woodward [WW5] we constructed a symplectic category \( \text{Symp}^# \) that contains all Lagrangian correspondences as composable morphisms. A Lagrangian correspondence \( M_0 \xrightarrow{L_{01}} M_1 \) between symplectic manifolds \( (M_0, \omega_0) \) and \( (M_1, \omega_1) \) is a Lagrangian submanifold \( L_{01} \subset M_0^- \times M_1 := (M_0 \times M_1, -\omega_0 \oplus \omega_1) \) in the product. Special cases are graphs of symplectomorphisms. Weinstein [W] generalized the algebraic composition of symplectomorphisms to the geometric composition of correspondences \( M_0 \xrightarrow{L_{01}} M_1 \) and \( M_1 \xrightarrow{L_{12}} M_2 \), given by the Lagrangian

\[
L_{01} \circ L_{12} := \pi_{M_0 \times M_2}((L_{01} \times L_{12}) \cap (M_0 \times \Delta M_1 \times M_2)) \subset M_0^- \times M_2.
\]

However, even after generic perturbation, this Lagrangian is generally only immersed. We make

\[
(1)
\]

the composition well defined by defining it as algebraic concatenation, and allowing sequences of Lagrangian correspondences as morphisms; e.g. the composition of \( L_{01} \) and \( L_{12} \) is the morphism from \( M_0 \) to \( M_2 \) given by the sequence \( M_0 \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2 \). One can now introduce an equivalence relation on these generalized Lagrangian correspondences, generated by

\[
(2)
(\ldots \to M_0 \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2 \to \ldots) \sim (\ldots \to M_0 \xrightarrow{L_{01} \circ L_{12}} M_2 \to \ldots)
\]

whenever the geometric composition \((1)\) is a transverse intersection and embedded by \( \pi_{M_0 \times M_2} \). So far, this is a geometrically meaningless algebraic trick. However, it will provide deep geometric insights by its interpretation as adiabatic limit in moduli spaces of a novel type of elliptic PDE that we introduced in [WW3]: A pseudoholomorphic quilt is a collection \( (u_i : \Sigma_i \to M_i)_{i=1,\ldots,N} \) of \( J_i \)-holomorphic maps from Riemann surfaces \( (\Sigma_i, j_i) \) to symplectic manifolds \( M_i \) which satisfy seam conditions as follows. Just as Lagrangian submanifolds are elliptic boundary conditions for pseudoholomorphic maps, Lagrangian correspondences are the natural elliptic seam conditions for pseudoholomorphic quilts: A seam between two boundary components \( C \subset \partial \Sigma_i \) and \( C' \subset \partial \Sigma_{t} \) is a diffeomorphism \( \sigma : C \to C' \), and the seam condition associated to a Lagrangian correspondence \( L_{it} \subset M_i^- \times M_t \) is \( (u_i \times u_t \circ \sigma)(C) \subset L_{it} \). For visual representations of quilts see e.g. [katrin/slides/MSRI1.pdf]. These form nonlinear elliptic PDEs since the seam conditions are locally equivalent to a Lagrangian boundary condition \( v(C) \subset L_{it} \) for a \( (-J_i \oplus J_t) \)-holomorphic map \( v : C \times [0, \varepsilon) \to M_i^- \times M_t \) induced by \( u_i \) and \( u_t \). In [WW2, WW3] we use pseudoholomorphic quilts consisting of strips to obtain a first symplectic invariant – quilted Floer homology, a generalization of Lagrangian Floer homology \( HF(L, L') \) – that is well defined under equivalences \((2)\). In a special case this result can be phrased as proven in [WW1].

Theorem 1. Let \( M_0, M_1, M_2 \) be compact (or exact) symplectic manifolds with the same monotonicity constant, and let \( L_0 \subset M_0, L_{01} \subset M_0^- \times M_1, L_{12} \subset M_1^- \times M_2, L_2 \subset M_2^- \) be compact, oriented, monotone Lagrangian submanifolds. If \( L_{01} \circ L_{12} \) is transverse and embedded as in \((2)\), and monotone, then there is a canonical isomorphism \( HF(L_0 \times L_1, L_0 \times L_2) \cong HF(L_0 \times L_2, L_0 \circ L_{12}) \).

This isomorphism is induced by an identification \( (L_0 \times L_1) \cap (L_{01} \times L_2) \cong (L_0 \times L_2) \cap (L_{01} \circ L_{12}) \) of the generators of the Floer chain complexes. The Floer differential for \( (L_0 \times L_2, L_0 \circ L_{12}) \) counts triples of pseudoholomorphic maps from strips to \( M_0, M_1, M_2 \). In the standard definition all strips have equal width, but quilts allow us to vary the widths and prove that, with the middle width sufficiently small, these triples are in one-to-one correspondence with the pairs of pseudoholomorphic strips in \( M_0, M_2 \) defining the Floer differential for \( (L_0 \times L_2, L_0 \circ L_{12}) \). Analytically, this is an adiabatic limit of nonlinear elliptic boundary value problems parametrized by \( \delta > 0 \), for maps \( u_{012} = (u_0, u_1, u_2) : \mathbb{R} \times [0, 1] \to M_0 \times M_1 \times M_2 \) of fixed energy \( E > 0 \) given by

\[
\partial_s u_{012} + \left( \begin{array}{c}
\delta J_0(u_0) \\
-\delta^{-1} J_1(u_1) \\
J_2(u_2)
\end{array} \right) \partial_t u_{012} = 0,
\]

\[
u_{012}(s, 0) \in L_0 \times L_{12},
\]

\[
u_{012}(s, 1) \in L_0 \times L_1,
\]

\[
\int |\partial_s u_{012}|^2 = E.
\]
where the “scaled almost complex structure” satisfies $J_{012} \circ J_{012} = (\text{id}_{M_0}, \delta^{-2}\text{id}_{M_1}, \text{id}_{M_2})$. Its $\delta \to 0$ adiabatic limit is the boundary value problem for maps $v_{02} = (v_0, v_2) : \mathbb{R} \times [0, 1] \to M_0 \times M_2$, $v_0(s, 0) \in L_0$, $\partial_s v_{02} + \left(J_{012}(v_0)\right)\partial_t v_{02} = 0$, $(v_0(s, 1), v_2(s, 0)) \in L_{01} \circ L_{12}$, $\int |\partial_s v_{02}|^2 = E$.

The core of Theorem 1 is a bijection, for small $\delta > 0$, between the spaces of Fredholm index 1 solutions with fixed energy $E$. The proof refines pseudoholomorphic curve techniques to e.g. control the domain dependence of estimates and required a new technique to exclude a blowup in the energy density $|\partial_s u_{012}|^2$. While classical techniques argue by finding holomorphic disks resp. spheres in the limit of a rescaling procedure, the seam conditions may yield a novel figure eight bubble given by a tuple of pseudoholomorphic maps $w_0 : \mathbb{R} \times (-\infty, -\frac{1}{2}) \to M_0$, $w_1 : \mathbb{R} \times [-\frac{1}{2}, \frac{1}{2}] \to M_1$, $w_2 : \mathbb{R} \times [\frac{1}{2}, \infty) \to M_2$ with seam conditions $(w_0(s, -\frac{1}{2}), w_1(s, -\frac{1}{2})) \in L_{01}$, $(w_1(s, \frac{1}{2}), w_2(s, \frac{1}{2})) \in L_{12} \quad \forall s \in \mathbb{R}$.

The seams of this quilt are the image, under stereographic projection, of two circles on $S^2$ that are tangent at the north pole – a conformal structure for which not even elliptic regularity is in reach of standard techniques. However, I was able to prove “$\epsilon$-regularity” by a combination of nonlinear mean value inequalities from [W6] and analysis of a degenerate limit case: If $|\partial_s u_{012}|^2$ is unbounded, then at least a universal minimal amount of energy $\epsilon > 0$ concentrates at some point.

In monotone settings (where energy and Fredholm index are positively proportional), this allows to exclude the blowup in low index moduli spaces. This provides the analytic basis for the following further developments:

- Theorem 1 and a generalization of Seidel’s long exact sequence [Sc] in [WW9] simplify various calculations of Floer homology and thus proofs of non-displacement results for Lagrangians under Hamiltonian diffeomorphisms, which were previously based on explicit identification of holomorphic curves. Novel calculations require the extension to multiply covered composition and settings beyond monotonicity in [W10]. This yields e.g. a confirmation of mirror symmetric calculations [Au] for the Chekanov torus and a product torus in $S^2 \times S^2$.

- The extension of Theorem 1 to “locally cleanly intersecting” compositions is within reach based on a Gromov compactness theorem for strip shrinking, including a removal of singularity and elliptic estimates for figure eight bubbles [BW] achieved in joint work with my graduate student Nate Bottman. However, the gluing analysis indicates that figure eight bubbling has smaller codimension than both disk and sphere bubbling, thus yields obstruction terms in the relation between the Floer complexes, which we plan to rigorously define by a polyfold description for moduli spaces of pseudoholomorphic quilts. This will also provide a framework for the orientation scheme from [WW8] for pseudoholomorphic quilts and quilted relative spin structures.

- In [WW5] we associate a functor on Donaldson-Fukaya categories of generalized Lagrangian submanifolds $\Phi_{L_01} : \text{Don}^\#(M_0) \to \text{Don}^\#(M_1)$ to any Lagrangian correspondence $L_{01} \subset M_0 \times M_1$ and prove that “categorification commutes with composition”, $\Phi_{L_01} \circ \Phi_{L_{12}} \cong \Phi_{L_{01} \circ L_{12}}$, if the geometric composition is embedded. More generally, we construct a symplectic 2-category which contains all Lagrangian correspondences as morphisms, and a categorification 2-functor. In [MWW] we outline the extension to chain level, which yields an $A_\infty$ functor on extended Fukaya categories $\text{Fuk}^\#(X^- \times Y) \to \text{Fuk}(\text{Fuk}^\#(X), \text{Fuk}^\#(Y))$ that is the symplectic analogue of a quasi-equivalence $D^b_{\text{coh}}(X \times X) \simeq \text{Fuk}(D^b_{\text{coh}}(X), D^b_{\text{coh}}(X))$ for certain derived categories of coherent sheaves on a projective variety $X$. This categorical structure forms an algebraic tool in

---

1 The mathematical notion of an adiabatic limit was introduced by Witten as deformation of a PDE by degeneration of a metric in certain directions. An important analytic feature is the preservation of an energy along the deformation.
the understanding of Fukaya categories and relations to algebraic geometry via Mirror Symmetry, which is being used to great effect by e.g. [Na, AS, AB, FSS, RS], most notably to obtain generation criteria for Fukaya categories. A polyfold setup for quilts with tangencies of seams based on [BW] will allow us to construct these functors for general symplectic manifolds.

2. Floer field theory

The compositability of morphisms in the symplectic category Symp and the categorification functor Symp \to \text{Cat} arising from its 2-category structure provide a general framework for constructing topological invariants, or category valued topological quantum field theories (TQFT). In fact, the attempt to construct a symplectic analogue of Floer’s gauge theoretic invariant for 3-manifolds [F] lead to the development quilted Floer homology, and is now possible along the lines of the following example from [WW6] based on [GWW]. Throughout, we must restrict to the category of compact, connected, oriented manifolds (thus our TQFTs will not be multiplicative).

To a surface Σ we associate the moduli space \( M(\Sigma) \) of semistable vector bundles of fixed coprime rank \( r \) and degree \( d \), with fixed determinant. (Equivalently, \( M(\Sigma) \) is a subspace of \( U(r) \) representations of \( \pi_1(\Sigma \setminus \{pt\}) \) with holonomy \(-1\) around the puncture.) To a cobordism \( Y \) between surfaces \( \partial Y = \Sigma_\pm \cup \Sigma_\pm \) one could associate a Lagrangian correspondence \( L(Y) \subset M(\Sigma_-) \times M(\Sigma_+) \) given by the restrictions of bundles to \( Y \) to the boundary (corresponding to \( U(r) \) representations of \( \pi_1(Y \setminus I) \) with holonomy \(-1\) around a line \( I \) connecting the punctures in \( \Sigma_\pm \)), but in general, \( L(Y) \) fails to be smooth. However, for simple cobordisms \( Y \) (that support a Morse function with at most one critical point), \( L(Y) \) is smooth and monotone. So for a closed 3-manifold \( Y \) we can use a cyclic decomposition \( Y = Y_{01} \cup \Sigma_1 \ldots \cup \Sigma_{k-1} Y_{(k-1)k}/\Sigma_k \sim \Sigma_0 \) along level sets \( \Sigma_i \) of a Morse function \( f : Y \to S^1 \) separating the critical points, to obtain a cyclic generalized Lagrangian correspondence

\[
M(\Sigma_0) \xrightarrow{L(Y_{01})} M(\Sigma_1) \to \ldots \to M(\Sigma_{k-1}) \xrightarrow{L(Y_{(k-1)k})} M(\Sigma_k) = M(\Sigma_0).
\]

To prove that its quilted Floer homology \( \text{HF}(L(Y_{01}), \ldots, L(Y_{(k-1)k})) \) is a topological invariant of \( Y \) (and the homotopy class \([f]\)) we note that any two decompositions of \( Y \) are related by a sequence of Cerf moves (cancellation and change of order in critical points). Each of these can be described in terms of cobordism splittings of the form \( Y_{ij} \cup Y_{jk} = Y_{ik} \), where \( Y_{ik} \) contains at most 2 critical points and \( L(Y_{ik}) = L(Y_{ij}) \circ L(Y_{jk}) \) is the transverse, embedded geometric composition of the two correspondences associated to the pieces. Theorem [4] then implies invariance,

\[
\text{HF}(\ldots, L(Y_{ij}), L(Y_{jk}), \ldots) \cong \text{HF}(\ldots, L(Y_{ij}) \circ L(Y_{jk}), \ldots) = \text{HF}(\ldots, L(Y_{ik}), \ldots).
\]

Similarly, we construct functor-valued invariants for cobordisms containing tangles and certain trivalent graphs [WW7] that satisfy exact triangles [WW9]. This provides symplectic versions of knot invariants similar to the combinatorial Khovanov-Rozansky homology [KR] and the gauge theoretic Kronheimer-Mrowka knot invariants [KM]. This quilt framework is also being applied by e.g. [Au2, L, R] to other gauge theoretic reductions such as Heegard Floer theory [OS].

More abstractly, this framework is an axiomatic system for constructing a functor from the (2+1)-dimensional connected bordism category to the monotone symplectic category, called 2+1 topological symplectic field theory (TSFT). Based on a classification of “Morse 2-functions” \( f : X^4 \to S^2 \) on 4-manifolds (for connected fibers by [GR]), I moreover outlined in [W11] that any 2+1 TSFT, under a single quilt axiom, induces a 2+1+1 TSFT, i.e. a 2-functor from the 2+1+1 bordism category to the symplectic 2-category. Composition with the categorification 2-functor then provides category-valued 2+1+1 topological field theories, in particular 4-manifold invariants.

This provides a link to Perutz’ Lagrangian matching invariants [P] that independently proposed a quilt-type symplectic analogue of Seiberg-Witten invariants. In particular, the relation of Cerf and quilt moves provides a symplectic approach to proving invariance of Perutz’ construction. Finally, the quilt axiom for 2+1+1 TSFTs is an equality of two quilt invariants arising from simple string diagrams in the 2-category. In a graphical interpretation by Schommer-Pries of Lurie’s topological
field theory, it corresponds to an axiom for fully dualizable field theories that is generally not satisfied in the symplectic 2-category, so it seems that the analysis of the strip shrinking can in some weak sense replace deep algebraic assumptions of the bordism hypothesis.

3. Relations between gauge theory and pseudoholomorphic curves

Donaldson’s 4-manifold invariants can be coupled with Floer’s instanton homology for 3-manifolds to form a gauge theoretic 3+1 topological field theory [D2], which was conjectured by Atiyah and Floer to have symplectic analogues arising from representation spaces of SU(2)-connections. Dostoglou-Salamon [DS] proved a first relation for 3-dimensional mapping tori to a symplectic Floer theory that involves no boundary conditions. My long term project is to establish similar relations for decompositions of closed 3-manifolds that lead to boundary conditions for holomorphic curves. For that purpose, I introduced Lagrangian boundary conditions for anti-self-dual connections on 4-manifolds with fibered boundary. In the example of a trivial SU(2)-bundle over $S^1 \times Y$, this yields a nonlocal PDE for connections $\Xi : S^1 \to \Omega^1(Y;\mathrm{su}(2))$,

$$
\partial_s \Xi + *F_\Xi = 0, \quad \Xi(s)|_{\partial Y} \in L \quad \forall s \in \mathbb{R},
$$

where $L \subset \Omega^1(\partial Y;\mathrm{su}(2))$ is a Lagrangian submanifold in the symplectic Hilbert space of $L^2$-connections, on which the Hodge $* \sigma$ operator of any metric on $\partial Y$ induces a complex structure. Gauge invariance of the equation forces the Lagrangian $L$ to lie in the subspace of flat connections as preimage of certain “Lagrangian subsets” of the singular SU(2)-representation space of $\pi_1(\partial Y)$. This means that the boundary condition for $A(s) := \Xi(s)|_{\partial Y}$ is given by a slice-wise first order equation $dA + A \wedge A = 0$ together with $3(\text{rk} H_1(\partial Y) - 1)$ real valued conditions on the holonomies of $A(s)$. As in the closed case, the anti-self-duality of $\Xi$ viewed as connection on $S^1 \times Y$, together with a gauge fixing condition, yields a nonlinear first order expression for $\Delta \Xi = d^*d\Xi + dd^*\Xi$, and together with Uhlenbeck compactness [U] this provides uniform bounds on all derivatives on the complement of finitely many bubbling points (where energy concentration can be described by instantons on $S^4$ bubbling off). However, the boundary conditions are not elliptic for a Laplace equation. Instead, I combined estimates for Cauchy-Riemann operators on infinite dimensional spaces such as $\partial_s + *\partial_t$ for maps to $\Omega^1(\partial Y;\mathrm{su}(2))$ (as in (5) below) from [W2] with refinements of gauge theoretic techniques such as Uhlenbeck compactness on manifolds with boundary [W1] to establish elliptic regularity, $\epsilon$-regularity, and removable singularity for (4) in [W3, W4].

In joint work [SW1] with Dietmar Salamon we built on this analytic framework to construct an instanton Floer homology for 3-manifolds with boundary and Lagrangians in the SU(2)-representation spaces of the boundary components. Finally, we can use abstract rather than topologically constructed Lagrangians, by replacing a borderline Sobolev estimate based on [HL] with a version of the div-curl lemma from harmonic analysis in [MrW]. In this process, I also obtained partial analytic results towards relations to closed instanton Floer homology and symplectic Floer homology with Lagrangian boundary conditions as outlined in [W5], but did not attempt to define Floer theory in singular symplectic spaces. In discussing possible nonsingular representation theoretic settings with Chris Woodward, we then asked ourselves whether topological invariance could be proven – without an Atiyah-Floer conjecture – in the symplectic category, and thus developed Floer field theory and quilts. However, I am now using this language to formulate more general (3+1)-dimensional quilted Atiyah-Floer conjectures:

We may study e.g. closed 4-manifolds $X$ by considering Morse 2-functions $f : X \to S^2$ and an SO(3)-bundle that restricts nontrivially to the fibers of $f$. These induce a quilted surface consisting of patches, seams, and punctures supporting topological string diagrams (tsd) in three different 2-categories, corresponding to three different moduli spaces of PDEs. Here a tsd is a generalization of string diagrams in 2-categories, namely an association of an object to each patch, a morphism to each seam, and a 2-morphism to each puncture. The 2-categories and associated PDEs are
the $(2+1+1)$-dimensional bordism category so that the tsd represents $X$ and thus the moduli space of anti-self-dual connections on $X$;

- the symplectic 2-category so that the tsd represents the data of domains and targets for a moduli space of pseudoholomorphic quilts;

- a mixed “instanton symplectic 2-category” in which the tsd represents data for a moduli space of anti-self-dual instantons with Lagrangian seam conditions.

To begin, we note that the singular set $\text{Sing} := f(\text{Sing} f) \subset S^2$ is a smooth 1-submanifold with the exception of finitely many nodes $N \subset \text{Sing}$ and crossings $C \subset \text{Sing}$. Now the diagrams are obtained as follows (see page 4 of [link to image for images]):

- patches $S_i \subset S^2$ are given by connected components of the smooth locus $S^2 \setminus \text{Sing}$ of the fibration, partially closed by the images of seams below, but with punctures at the nodes and crossings;

- the topological object associated to a patch is a smooth fiber $\Sigma^i$;

- the symplectic object associated to a puncture is a canonical quilted Floer class $\lambda_p \in \text{HF}(L_p)$ induced by the cyclic tuple $L_p$ of Lagrangian correspondences associated to the seams with ends on $p$. These are sequences as in (3) arising from the Morse function $f : \partial X_p \to \partial D$.

A canonical instanton 2-morphism associated to a puncture is the relative Donaldson invariant of $X_p$ in the instanton Floer homology $\text{HF}^{\text{inst}}(\partial X_p)$ of the closed (smooth) 3-manifold. However, it needs to get transferred to an element of the quilted instanton Floer homology of the cyclic tuple $L_p$ of Lagrangian correspondences associated to the puncture.

This string diagram description now allows to localize the adiabatic limit analysis. In the interior of patches with local coordinates $(s, t)$, degeneration of the metric along the fibers by $\text{d}s^2 + \text{d}t^2 + \varepsilon^2 g_{\Sigma}$, by the analysis of [DS], deforms the anti-self-duality equation on $S \times \Sigma$ to the Cauchy-Riemann equation for maps $S \to M(\Sigma)$. Possible bubbles are instantons on $S^4$, holomorphic spheres in $M(\Sigma)$, and the intermediate case of instantons on $\mathbb{C} \times \Sigma$, for which I proved integrality of the charge in [W7]. The latter suggests a partial removal of singularity to a stable holomorphic bundle on $\mathbb{C}P^1 \times \Sigma$, as established by Biquard-Jardim [BJ] for $\Sigma = T^2$ under quadratic curvature decay assumptions. One day I hope to understand the natural curvature decay implied by finite energy. Before that, however, I hope to complete the adiabatic limit analysis near a seam $C_\sigma$ for instantons with Lagrangian seam conditions, or equivalently a boundary value problem for triples $A : (-1, 1) \times [0, 2) \to \mathcal{A}(\Sigma)$ and $\Phi, \Psi : (-1, 1) \times [0, 2) \to \Omega^0(\Sigma, \mathfrak{su}(2))$ with $\varepsilon \to 0$,

\begin{equation}
\begin{cases}
\partial_s A - d_A \Phi + \ast (\partial_t A - \ast d_A \Psi) = 0, \\
\partial_s \Psi - \partial_t \Phi + [\Phi, \Psi] + \varepsilon^{-2} \ast F_A = 0,
\end{cases}
\end{equation}

$A(\cdot, 0) \in \mathcal{L}$, \quad $\int |\partial_s A - d_A \Phi|^2 + \varepsilon^{-2} |F_A|^2 = E$.  

\[ \Box \]
This splitting clearly shows the Cauchy-Riemann operator \( \partial_s A + \ast \partial_t A \) up to infinitesimal gauge actions \( d_A \Phi, d_A \Psi \). Using a combination of 2- and 4-dimensional mean value inequalities from [W6], I can obtain uniform bounds on the energy density \( |\partial_s A - d_A \Phi|^2 + \varepsilon^2 |F_A|^2 \) on the complement of finitely many bubbling points, which forces the limit connections \( A(s, t) \) to be flat. It remains to finalize this limit by a combination of [DS] and [W3], reverse the limit by a gluing theorem, and apply another degeneration to relate anti-self-dual instantons on two sides of a shrinking cobordisms to anti-self-dual instantons with Lagrangian seam conditions. An alternative approach is to shrink the fibers \( \Sigma_i \) with volume \( \varepsilon^2 \) and cobordisms \( Y_\sigma \) with volume \( \varepsilon^3 \) simultaneously to directly relate ASD instantons to holomorphic quilts. Bubbling is controlled as above, but the limit argument essentially requires estimates for holomorphic maps with boundary values converging to a Lagrangian. One indication of substantial difficulties in the boundary conditions is the fact that [W4] can capture possible bubbling at the boundary not as instantons on a half space \( \mathbb{H}^4 \), as conjectured in [Sa], but as instantons on \( \mathbb{H}^2 \times \Sigma \) or even holomorphic half-planes in \( \mathcal{A}(\Sigma) \) with \( \mathcal{L} \) boundary values.

A similar relationship for the Seiberg-Witten 4- and 3-manifold invariants was established by Taubes [T1, T2] to a version of Gromov-Witten invariants and to embedded contact homology [Hu, HIS, H1] resp. Heegard Floer homology [CLT]. Recent work of my graduate student Tim Nguyen [Ng1, Ng2, Ng3] sets up Lagrangian boundary conditions for Seiberg-Witten equations and [Hu, HS, HT] resp. Heegard Floer homology [CLT]. Recent work of my graduate student Tim Taubes [T1, T2] to a version of Gromov-Witten invariants and to embedded contact homology [Hu, HS, HT] resp. Heegard Floer homology [CLT]. Recent work of my graduate student Tim Taubes [T1, T2] to a version of Gromov-Witten invariants and to embedded contact homology [Hu, HS, HT] resp. Heegard Floer homology [CLT]. Recent work of my graduate student Tim Taubes [T1, T2] to a version of Gromov-Witten invariants and to embedded contact homology [Hu, HS, HT] resp. Heegard Floer homology [CLT]. Recent work of my graduate student Tim Taubes [T1, T2] to a version of Gromov-Witten invariants and to embedded contact homology [Hu, HS, HT] resp. Heegard Floer homology [CLT]. Recent work of my graduate student Tim Taubes [T1, T2] to a version of Gromov-Witten invariants and to embedded contact homology [Hu, HS, HT].

4. SYMPLECTIC NON-SQUEEZING IN INFINITE DIMENSIONS

The study of non-squeezing effects for flows of infinite dimensional Hamiltonian systems arose from discussions with Alija Barakat and Gigliola Staffilani of such results for compact perturbations of linear flows studied [K], the 1-dimensional cubic nonlinear Schrödinger equation [B], and the Korteweg-de Vries equation [CKSTT]. These were all based on finite dimensional approximation, thus allowing to make use of Gromov’s non-squeezing theorem [G] in finite dimensions. We aim to find an infinite dimensional symplectic point of view on these results by finding a set of axioms under which a symplectic embedding \( \phi : B_R \hookrightarrow H \), \( \phi^* \omega = \omega \) of a ball \( B_R := \{ x \in H \mid \| x \| < R \} \) in a symplectic Hilbert space \( (H, \omega) \) satisfies the non-squeezing property:

\[
\phi(B_R) \subset Z_{\pi, r} := \{ x \in H \mid \| \pi(x) \| < r \} \quad \implies \quad R \leq r,
\]

where \( Z_{\pi, r} \) is the cylinder over a symplectic 2-plane \( \pi(H) \subset H \) with orthogonal projection \( \pi : H \to H \).

An example of a squeezing map is the shift \( \phi((z_1, z_2, \ldots)) = (0, z_1, z_2, \ldots) \) on the sequence space \( \ell^2 = \{ \mathbf{z} = (z_i)_{i \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} \mid \sum_{j=1}^\infty |z_j|^2 < \infty \} \) with symplectic structure \( \omega(z, w) = \sum_{j=1}^\infty \text{Re}(z_j \overline{w_j}) \).

This can be excluded by the assumption that \( \phi \) has open image (as is the case for the flow of a Hamiltonian system). That also allows to implement Gromov’s strategy of proof. However, transferring the analysis of pseudo-holomorphic curves to the infinite dimensional ambient space \( H \) needs to overcome the noncompactness of the Sobolev embedding \( W^{1, 2}(S^2, H) \hookrightarrow L^2(S^2, H) \).

We propose to use a scale structure on \( H \) in the sense of Hofer-Wysocki-Zehnder [HWZ1]. For example, a scale structure on \( H = L^2(S^1, \mathbb{C}) \) is given by the sequence of Hilbert spaces \( H_k = W^{k, 2}(S^1, \mathbb{C}) \), whose embeddings \( H_k \hookrightarrow H_{k-1} \) are compact and dense for all \( k \in \mathbb{N} \). Now the embedding \( W^{1, 2}(S^2, H_0) \cap L^2(S^2, H_1) \hookrightarrow L^2(S^2, H_0) \) is nothing but the compact Sobolev embedding \( W^{1, 2}(S^2 \times S^1) \hookrightarrow L^2(S^2 \times S^1) \). If in fact \( H \) carries a scale symplectic structure \((H_k, \omega_k)\) and \( J_0 \) is a scale-compatible complex structure, i.e. \( \omega_k(\cdot, J_0 \cdot) \) defines a complete metric on each \( H_k \), we can then use this structure to obtain compactness properties for \( J_0 \)-holomorphic maps \( u : \Sigma \to H_\infty := \bigcap_{k=0}^\infty H_k \) from bounds on the energies \( \int u^* \omega_k = \frac{1}{2} \int \| du \|^2_{H_k} \) that result from fixing the homotopy class. In the above example, \( J_0 = i \) is compatible with the real inner products on \( W^{k, 2}(S^1, \mathbb{R}^2) \). The frequency spaces in [B, CKSTT] carry similar canonical structures. However, the compactness arguments require scale-compatibility of \( \phi_* J_0 \), which is not entirely unrealistic since the flows \( \phi \) are known to be scale-smooth in these cases, in fact smooth on each scale \( H_k \).
5. Regularization of pseudoholomorphic curve moduli spaces

One of the central technical problems in the theory of pseudoholomorphic curves, which provides many of the modern tools in symplectic topology, is to construct algebraic structures by extracting homological information (e.g. a count) from moduli spaces of pseudoholomorphic curves. We will refer to this technique as regularization and note that it requires two distinct components. On the one hand, some perturbation technique is used to achieve coherent transversality, which gives each moduli space a differentiable structure that (together with compactness) induces a count or a chain. These perturbations need to be made coherently on a variety of moduli spaces, so that boundary points of higher dimensional moduli spaces are identified with pairs of points in lower dimensional ones. This provides relations between the counts or chains of the different moduli spaces, which then form an algebraic structure such as a chain complex or an $A\infty$-algebra. To prove that these structures are independent from the choices involved in achieving transversality (e.g. that the homology of the chain complex is well defined) requires the second component, invariance, which is often achieved by cobordism techniques.

5.1. Moduli spaces of unparametrized pseudoholomorphic maps. A first example of pseudoholomorphic curve invariants are the spherical Gromov–Witten invariants $\langle \alpha_1, \ldots, \alpha_k \rangle_{0,A} \in \mathbb{Q}$ of a symplectic manifold $(M, \omega)$. They are defined as generalized count of $J$-holomorphic curves of genus 0 (i.e. spheres) in class $A \in H_2(M)$ that meet $k$ representing cycles (e.g. submanifolds) of homology classes $\alpha_i \in H_*(M)$. This number should be independent of the choice of $J$ (in the contractible space of $\omega$-compatible almost complex structures on $M$), and of the cycles representing $\alpha_i$. A special case are complex structures given by $J = i$ for local coordinates of $M$ in $\mathbb{C}^n$, for which one can work in the algebraic setting that describes $J$-holomorphic curves as the zero locus of holomorphic functions on $M$. In general symplectic manifolds, the almost complex structures $(J : TM \to TM$ satisfying $J^2 = -\text{id})$ cannot be made constant in local coordinates and hence do not allow for this algebraic approach. The approach introduced by Gromov [G] is to describe pseudoholomorphic curves as maps to $M$ that satisfy the Cauchy–Riemann PDE, modulo regularizing the counts of pseudoholomorphic spheres $\langle \rangle_{0,A} \in \mathbb{Q}$ in class $A$, the Gromov–Witten moduli space $\mathcal{M}(A, J) := \hat{\mathcal{M}}(A, J)_{/G}$ is constructed from the solution space

$$\hat{\mathcal{M}}(A, J) := \{ u : S^2 \to M \mid u_\ast S^2 = A, \bar{\partial}_J u = 0 \}$$

as quotient by the reparametrization action of the Möbius group $G = \text{PSL}(2, \mathbb{C})$,

$$\text{PSL}(2, \mathbb{C}) \times \hat{\mathcal{M}}(A, J) \to \hat{\mathcal{M}}(A, J), \quad (\gamma, u) \mapsto u \circ \gamma.$$

Here the Cauchy–Riemann operator is given by $\bar{\partial}_J u = \partial_u J + J(u) \partial_{\bar{u}} u$ in local coordinates. It is elliptic but nonlinear since the almost complex structure $J$ varies with the base point. The moduli space $\mathcal{M}(A, J)$ is contained in the compact moduli space $\hat{\mathcal{M}}(A, J)$ formed by the equivalence classes of $J$-holomorphic genus 0 nodal curves in $M$. Here e.g. the curves with a single node are described as fiber product $\{ (f, g) \in \hat{\mathcal{M}}(B, J) \times \hat{\mathcal{M}}(A - B, J) \mid f(0) = g(0) \}$ modulo the automorphism group of reparametrizations fixing the marked points $0 \in S^2$.

More abstractly, pseudoholomorphic curve moduli spaces have Fredholm descriptions roughly as follows: The Cauchy–Riemann operator $\sigma := \bar{\partial}_J : B \to E$ is a Fredholm section of a Banach bundle $E \to B$ of completions of spaces of smooth maps, e.g. $B = \text{cl}_{W^{1,p}} \{ u \in C^\infty(S^2, M) \mid u_\ast S^2 = A \}$. This section is equivariant under a finite dimensional Lie group $G$ acting by reparametrizations, e.g. $G \times B \to B, (\gamma, u) \mapsto u \circ \gamma$. Although this action is nowhere differentiable, its zero set $\hat{\mathcal{M}} = \sigma^{-1}(0)$ is invariant under $G$, and the restricted action is smooth. The moduli space $\mathcal{M} = \sigma^{-1}(0)_{/G}$ is typically noncompact but has a “compactification” $\overline{\mathcal{M}}$, which is a compact Hausdorff space given by adding singular (“broken” or “nodal”) curves to $\mathcal{M}$. (Unless $\mathcal{M}$ is cut out transversely by $\sigma$,
however, it might not be dense in $\mathcal{M}$.) From this perspective, regularizations of pseudoholomorphic curve moduli spaces require a highly nontrivial generalization of the finite dimensional regularization based on Sard’s and the implicit function theorem.

**Finite Dimensional Regularization Theorem:** Let $s : B \to E$ be a smooth section of a finite dimensional vector bundle such that $s^{-1}(0) \subset B$ is compact. Then there exists a compactly supported, smooth perturbation $p : B \to E$ such that $s + p$ is transverse to the zero section, and hence $(s + p)^{-1}(0)$ is a smooth manifold. Moreover, the perturbed zero sets for any two such perturbations are cobordant.

5.2. **Geometric Regularization.** For some classes of symplectic manifolds and moduli spaces, regularization can be achieved by a geometric approach. It proceeds in two steps that do not generalize the finite dimensional regularization, but rather make use of special geometric properties of curves in these spaces. Firstly, using a weak injectivity property of pseudoholomorphic maps, one obtains transversality $\sigma + p = \partial J' \pitchfork 0$ by perturbing the almost complex structure $J' \approx J$ (or adding other geometrically motivated differential operators) while preserving the $G$-symmetry, although even in finite dimensions equivariant transverse perturbations exist only for very special group actions. Secondly, if the singularity formation is controlled geometrically, then although the unperturbed zero set is far from compact, the perturbed zero sets for different choices of $J'$ are cobordant in the sense that their Gromov-compactifications are cobordant. While the classes of symplectic manifolds in which this approach applies are rather special, they include most natural examples of special interest, such as tori, complex projective spaces, cotangent bundles, and further classes of topologically simple symplectic manifolds. So much progress in symplectic geometry and its applications to low dimensional topology was made by using geometrically regularized moduli spaces of pseudoholomorphic curves. For example, regularization for the moduli spaces of quilted Floer trajectories described Section 1 can be obtained by perturbation of the almost complex structures. However, our proof [WW2] overlooked possible injectivity failure, in which one strip is constant and an adjacent one is nonconstant so that the linearized seam conditions vary. It took five increasingly technical applications of Sard’s theorem in [WW3] to exclude this regularity obstruction by further geometric perturbations of the Lagrangian and the almost complex structure.

This leaves little hope for geometric techniques to regularize moduli spaces of pseudoholomorphic quilts that involve e.g. figure eight bubbling – similar to the SFT moduli spaces of $\ell GHI$, for which geometric regularization is impossible in any symplectic manifold. Moreover, for any moduli space in a general symplectic manifold, no perturbation as PDE with Cauchy-Riemann symbol can provide regularization. This is since none such perturbation is visible after the rescaling used to describe the singularity formation as bubbling off of e.g. pseudoholomorphic spheres, and these generally don’t have sufficient injectivity properties. Hence regularization schemes for general pseudoholomorphic curve moduli spaces need to work more abstractly, similar to the finite dimensional regularization. In particular, invariance now crucially requires that transversality is achieved directly on the compactified moduli space. Contrary to the geometric approach, this requires to deal with two analytic difficulties – dividing by the automorphism group and gluing constructions in the compactification – before achieving transversality. The first instance in which in the 1990s novel abstract techniques were developed was the case of nowhere injective spheres, which are then multiply covered and have nontrivial finite symmetry groups (“isotropy”). So the development of abstract regularization approaches was focussed on dealing with isotropy by orbifold methods, while analytic issues were dealt with in less detail, and – as recently exhibited in my work with Dusa McDuff [MW1] – some severe topological issues were entirely overlooked. The following gives a rough description of the three abstract approaches currently used in symplectic topology, some of their basic foundational issues, and my work on the foundations.

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2A geometric approach [CM, I] using nonlocal PDE perturbations applies to some moduli spaces in general symplectic manifolds, but is not universally applicable since e.g. it cannot regularize SFT moduli spaces.
5.3. **Regularization by global obstruction bundles.** This approach by e.g. [LiT] and [Si] aims to extend successful techniques from algebraic geometry and gauge theory, which generalize the finite dimensional regularization to finite dimensional bundles over infinite dimensional spaces. For application to pseudoholomorphic curve moduli spaces, the basic idea is to use the Fredholm property of the Cauchy-Riemann operator to construct a finite dimensional obstruction bundle over an infinite dimensional space of (generalized) maps modulo reparametrization, which contains the compactified moduli space. The foundational difficulty here is that – unlike the smooth action of the (infinite dimensional) gauge group on Sobolev spaces of connections – the action of reparametrization groups (though finite dimensional) in (9) does not extend to a differentiable action on any known Banach space completion of spaces of smooth maps. This causes differentiability failure in the relation between local charts of [LiT] and the survey by McDuff [M], which is irreplaceable for this approach. A more promising approach in [Si] of replacing a global differentiable structure by more algebraic topology was never published due to another analytic gap.

In recent discussions with Dietmar Salamon, we moreover debunked the following folk understanding of how obstruction bundles might be constructed in concrete applications. This might serve as illustration of how the obstacles to regularization often defy all intuition.

**Remark 2.** Let $H$ be a Banach space and $(V_i(s) \subset H)_{s \in [0,1]}$ for $i = 1, 2$ two smooth families of finite dimensional subspaces. Then it is not necessarily true that there exists a smooth family $W(s) \subset H$ of finite dimensional subspaces such that $V_i(s) \subset W(s)$ for $i = 1, 2$. In fact, there exists a smooth family of unit vectors in the sequence space $v(s) \in \ell^2$ such that any continuous family of subspaces $W(s) \subset \ell^2$ which contain both $v(0)$ and $v(s)$ is infinite dimensional.

5.4. **Kuranishi Regularization.** This approach was introduced in the mid 1990s by [FO] and implicitly by [LiT]. (Its further development in [J] got withdrawn recently.) It aims to generalize the finite dimensional regularization theorem to Kuranishi spaces, which have local descriptions as zero sets of smooth sections in finite dimensional bundles. These finite dimensional reductions arise from local choices of obstruction spaces. This was understood to resolve the differentiability issue of the reparametrization action by the reduction to finite dimensional spaces of smooth maps. However, we encountered the same issue again when working on an explicit construction of sum charts (arising from a sum of obstruction spaces) in [MW1], by which compatibility is established. We found that the only resolution is to work with special – geometrically constructed – obstruction spaces. This was understood to resolve the differentiability failure in the finite dimensional regularization theorem to Kuranishi spaces, which have local descriptions implicitly by [LiT]. (Its further development in [J] got withdrawn recently.) It aims to generalize the finite dimensional regularization theorem to Kuranishi spaces, which have local descriptions as zero sets of smooth sections in finite dimensional bundles. These finite dimensional reductions arise from local choices of obstruction spaces. This was understood to resolve the differentiability issue of the reparametrization action by the reduction to finite dimensional spaces of smooth maps. However, we encountered the same issue again when working on an explicit construction of sum charts (arising from a sum of obstruction spaces) in [MW1], by which compatibility is established. We found that the only resolution is to work with special – geometrically constructed – obstruction spaces. This was understood to resolve the differentiability failure in the finite dimensional regularization theorem to Kuranishi spaces, which have local descriptions implicitly by [LiT]. (Its further development in [J] got withdrawn recently.)

**Theorem 3.** Let $\mathcal{M}(A, J)$ be a compact moduli space of simple $J$-holomorphic maps $S^2 \to M$ in class $A$. There exists an open cover $\mathcal{M}(A, J) = \bigcup_{i=1}^{N} F_i$ by footprints of basic Kuranishi charts $(K_i)_{i=1,\ldots,N}$. If the obstruction spaces of charts with overlapping footprints satisfy an additivity condition, then for any tuple $(K_i)_{i \in I}$ of basic charts there exists a sum chart $K_I$ with obstruction space $\prod_{i \in I} E_i$ and footprint $F_I = \bigcap_{i \in I} F_i \subset \mathcal{M}(A, J)$, and a coordinate change $\Phi_{i|I}$ from a restriction $K_i|F_i$ of each basic chart to $K_I$. Finally, there exist further coordinate changes $\Phi_{I|J}$ for any $I \subset J$ that satisfy the cocycle condition $\Phi_{JK} \circ \Phi_{IJ} = \Phi_{IK}$ on their overlap of domains.

The second foundational part of this regularization approach is an abstract **Kuranishi regularization theorem**, which associates a rational cycle (or cobordism class of branched manifolds) to any Kuranishi space. It is in this part that we found the most surprising foundational issues. These were likely overlooked because they are clearly visible only in simplified settings of e.g. trivial isotropy, when one expects integer cycles or true manifolds, in particular a Hausdorff property. A second issue, whose subtlety was vastly underestimated, is the compactness requirement on the regularized space. In general, the Hausdorff resp. compactness property of the zero set of a section can

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A curve is simple if it is not multiply covered. For closed curves, this assumption ensures trivial isotropy.
be preserved under perturbations only when the ambient space is Hausdorff resp. locally compact. However, the natural ambient space arising from the domains of Kuranishi charts modulo transition maps is generally not Hausdorff, and locally compact only in trivial cases. In fact, the first natural example arising from two compatible Kuranishi charts with domains of different dimension even fails first countability, and hence metrizability. Even more fundamentally, the transition data between finite dimensional reductions that can be constructed in practice, e.g. as in Theorem 3, does not form a reflexive or transitive relation since the transition maps embed lower dimensional charts into higher dimensional ones, but the intersection of two such embeddings is generally not transverse. Some partial transversality – which we call additivity – can be achieved by appropriate choice of obstruction spaces. Moreover, compositions of these embeddings only satisfy the weak cocycle condition (on overlaps) so that the categorical setup – with the union of domains forming the space of objects and morphisms given by the transition maps – lacks well defined composition of morphisms. Though a regularization theorem seems unlikely in view of all these obstacles, hard technical wrestling amounting to 120 pages in [MW1] allows us to associate a virtual fundamental class to additive weak Kuranishi structures as follows.

**Theorem 4.** Let $\mathcal{K}$ be an oriented, $d$-dimensional, additive weak Kuranishi structure with trivial isotropy groups on a compact metrizable space $X$. Then $\mathcal{K}$ determines a cobordism class of smooth, oriented, compact manifolds, and an element $[X]^{\text{vir}}_{\mathcal{K}}$ in the Čech homology group $\check{H}_d(X; \mathbb{Q})$. Both depend only on the additive weak cobordism class of $\mathcal{K}$.

In the course of proving this theorem, we developed basic techniques of

- constructing a categorical structure from a finite number of noncompact finite dimensional manifolds and partially composable transition maps,
- achieving properness conditions that ensure the Hausdorff property of the realization (space of objects modulo morphisms),
- a finite dimensional regularization theorem in an ambient space $B$ that is Hausdorff but not locally compact or metrizable,
- a theory of orientations on Kuranishi structures, which necessitated some corrections to [MS],
- a cobordism theory which accommodates the fact that constructions of Kuranishi structures for moduli spaces of pseudoholomorphic curves can at best be cobordant in a weak Kuranishi sense.

Making our abstract regularization theorem applicable to general holomorphic curve moduli spaces requires two generalizations in [MW2]. Firstly, we are developing a groupoid version of the theory to allow for nontrivial isotropy groups. Secondly, relaxing the smoothness assumption on the structure maps in the Kuranishi category permits the construction of Kuranishi charts using well established gluing analysis from [MS]. Finally, the necessity of a priori transversality (additivity) is a novel requirement on Kuranishi charts, which we hope to achieve by simultaneous geometric perturbation of a finite number of charts. We are moreover incorporating gluing and nontrivial isotropy to provide a construction of an additive weak Kuranishi structure, and thus a virtual fundamental cycle, for spherical Gromov–Witten invariants of general symplectic manifolds in [MW3].

### 5.5. Polyfold Regularization

This approach is being developed in [H1, H2, HWZ0, HWZ1, HWZ2, HWZ3, HWZ4, HWZ5, HWZ6, HWZ11, HWZ12] since 2000 and aims to generalize the finite dimensional regularization to an infinite dimensional bundle over an infinite dimensional space, so that the compactified moduli space is the zero set of a smooth Fredholm section. It resolves the differentiability failure of reparametrization actions by replacing the notion of smoothness in Banach spaces by a notion of scale-smoothness which holds for reparametrizations. To implement this, the authors redeveloped linear as well as nonlinear functional analysis in the scale-smooth category. Furthermore, this approach interprets the pregluing construction as a chart map for an ambient space $\tilde{B}$ of generalized maps which contains the compactified moduli space. Injectivity of
pregluing fails dramatically but can be interpreted as generalization of a Banach manifold chart, where the usual model domains (open subsets of a Banach space) are replaced by relatively open subsets in the image of a scale-smooth retraction on a scale-Banach space. This makes it necessary to redevelop differential geometry in the context of retractions and scale-smoothness.

A second nontrivial part of this regularization approach is to cast a given moduli space as zero set of a polyfold Fredholm section, and to prove that the polyfold setup is unique up to cobordism in the polyfold category. While the foundational issues seem to have been overcome in this approach, a substantial amount of details for the application to Gromov-Witten invariants and Symplectic Field Theory announced in [HWZ7, HWZ8, HWZ9, HWZ10] are not yet available. Moreover, anyone who wishes to apply the polyfold regularization theorem needs to invest a substantial amount of energy into learning this radically new language from a vast body of literature. In following this development closely since 2004, one of my aims has been to make it accessible to a larger audience. Thus I have been giving a variety of expository lectures, organizing workshops, and taught a lecture course at MIT. In joint work with Joel Fish et al, I just finished an exposition of polyfold theory [FFGW] that on 60 pages summarizes the essential philosophy and notions in ca. 1000 pages of original literature. As part of this quest to simplify the abstract framework needed for applications, I introduced a simplified abstract Fredholm property in [WS], which enabled me to give a fairly concise proof of the [HWZ2] Fredholm property for Hamiltonian Floer theory – taking for granted the setup of charts and bundles that should appear in [HWZ9] – thus proving a special case of [HWZ10] which will be needed for all polyfold construction of Floer homology.

The final issue of polyfold regularization is the need to cast the basic variations of Cauchy-Riemann equations into the appropriate framework. To further this quest, I have formed polyfold lab by encouraging several graduate students and postdocs to work on the construction and application of polyfold Fredholm theory. Besides [BW], this is presently yielding new proofs/constructions of the Arnold conjecture and Fukaya $A_{\infty}$-category – both corner stones of symplectic topology, which are described in the following. It is anticipated that combinations of the building blocks from Gromov-Witten, SFT, and our work should allow to cast most other moduli spaces of pseudoholomorphic curves into a polyfold framework.

5.6. Arnold’s conjecture for Hamiltonian systems. Andreas Floer developed Floer homology to prove the weak Arnold conjecture: The number of periodic trajectories of a periodic Hamiltonian system is bounded below by the total rank of homology of the symplectic manifold. His proof [F3] uses a geometric regularization approach to argue with an $S^1$-symmetry in the case of a small autonomous Hamiltonian that the moduli spaces of pseudoholomorphic cylinders coincide with the spaces of Morse trajectories. Together with an invariance of Floer homology under Hamiltonian deformations, this implies an isomorphism between Floer homology and Morse homology, and thus a lower bound on the number of generators of the Floer complex – the periodic trajectories. Due to the challenges of sphere bubbling, Floer restricted his approach to aspherical symplectic manifolds, but it was quickly generalized to semipositive manifolds – the most general case in which sphere bubbling can be regularized by variations of $J$. Beyond that, in the language of Kuranishi structures, the idea for generalizing the symmetry part of this argument is that a Kuranishi space $X$ of virtual dimension 0, on which $S^1$ acts such that the fixed points $F \subset X$ are isolated solutions, should allow for a regularization given by the fixed points. However, almost no further details of this construction were published. Such a quotient theory may be within reach of the current polyfold technology, but has not yet been attempted.

An alternative approach to identifying Floer and Morse homology via explicitly constructed chain maps was proposed in [PSS] with a geometric regularization scheme that again requires $S^1$-equivariant transverse perturbations, and thus is limited to a small class of symplectic manifolds. In joint work with Peter Albers and Joel Fish [APW], we developed an approach to the weak Arnold conjecture for general symplectic manifolds by the PSS approach with an additional algebraic
argument that eliminates the need for $S^1$-equivariance. Our approach is to construct the PSS moduli spaces by combining the polyfold Fredholm property of SFT-Cauchy-Riemann operators (that is expected to be provided by [HWZ9, HWZ10]) with a smooth structure on generalized spaces of finite and semi-infinite Morse trajectories established in [W9]. Besides providing a rigorous proof of one of the pivotal conjectures for the field, we also plan to write the manuscript in an expository style such that it serves as demonstration of effective applications of polyfold techniques.

A proof that the PSS morphism is in fact an isomorphism between Morse and Floer homology will require an abstract notion of codimension 2 strata in polyfolds, and a proof that zero sets of polyfold sections of Fredholm index $\leq 1$ can be perturbed to avoid such strata. We hope to prove the necessary abstract refinement of the polyfold regularization, which in the application will yield polyfold constructions of a class of spectral symplectic invariants.

5.7. Pseudoholomorphic disks and the Fukaya category. Almost any question in symplectic geometry can be phrased in terms of Lagrangian submanifolds $L \subset M$, and these can be studied via pseudoholomorphic curves with Lagrangian boundary conditions. However, this nonlinear elliptic PDE allows, in addition to sphere bubbling, which in the polyfold framework is described as interior phenomenon, for disk bubbling at the boundary which yields boundary components of moduli spaces. This poses algebraic obstructions in the construction of Floer homology for pairs of Lagrangian submanifolds. Instead of the differential on the Floer complex satisfying $d \circ d = 0$, [FOOO] introduced a new algebraic framework of $A_\infty$-algebras associated to each Lagrangian, and generalized the Floer differential to a countable number of algebraic operations between the Floer complex and the $A_\infty$-algebras. On the one hand, this allows to construct algebraically twisted differentials, for which Floer homology is well defined. On the other hand, all Lagrangian submanifolds of a fixed symplectic manifold can be related by a further $A_\infty$-structure arising from moduli spaces of pseudoholomorphic polygons with boundary arcs in different Lagrangians. Together, this forms the Fukaya $A_\infty$-category of a fixed symplectic manifold, which in particular appears in the Mirror Symmetry conjectures and theorems relating symplectic geometry to algebraic geometry.

All of these algebraic structures need to be constructed from moduli spaces of pseudoholomorphic curves, and thus suffer from the regularization issues described previously. While a large body of work can rigorously build on geometric regularization for a class of symplectic manifolds that disallows bubbling as in [Se2], any general construction of the Fukaya category or even an $A_\infty$-algebra for a general Lagrangian submanifold $L \subset M$ requires abstract regularization techniques. A further difficulty is that the $A_\infty$-algebra proposed in [FOOO] is generated by a countable collection of currents on $L$ via an algebraic limit. A natural idea to remedy this unfortunate situation for calculations is to work on the Morse complex of $L$ that is generated by the finite number of critical points of a Morse function on $L$. Similar to the PSS morphisms, one can then study moduli spaces of pseudoholomorphic disks connected by Morse flow lines. In [W12] I proposed a construction of finitely generated $A_\infty$-algebras from such moduli spaces and worked out some ideas for their polyfold description in a graduate course at MIT. Jiayong Li, a graduate student founding member of polyfold lab, is now making rapid progress towards the construction of a finitely generated $A_\infty$-algebra for a general compact Lagrangian submanifold $L \subset M$ by polyfold methods.

By coupling this work with a polyfold setup for Lagrangian Floer theory and relative invariants, we plan to extend the construction of Fukaya categories in [Se2] to all symplectic manifolds. Finally, by coupling this with [BW] and a polyfold setup for quilted invariants we plan to obtain a full generalization of the symplectic 2-category [WW5] (which allows only monotone Lagrangian and symplectic manifolds) to a higher $A_\infty$-categorical structure that extends [MWW] and relates all geometrically bounded symplectic manifolds.
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