Dynamical invariants of categories associated to mapping tori

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Overview

1. Motivation
2. Construction of the mapping torus categories
3. Statement of the main theorem and the idea
4. A family of bimodules
5. Proof of the main theorem
Let $(M, \omega)$ be a symplectic manifold and $\phi$ be a symplectomorphism. Define the symplectic mapping torus as

$$\bar{T}_\phi = M \times \mathbb{R} \times S^1/(x, t, s) \sim (\phi(x), t + 1, s)$$

It is a symplectic manifold fibered over $T^2$. Assume $\phi$ is not Hamiltonian. **Question:** How can we distinguish $\bar{T}_\phi$ and $\bar{T}_{id_M} = M \times T^2$?

**Answer:** Assume $M$ is compact and $H^1(M) = 0$. We can try to use an invariant called the Flux group to distinguish them.
Given a compact symplectic manifold $X$, flux group is a discrete subgroup $\Gamma \subset H^1(X; \mathbb{R})$ which measures the abundance of loops/circles in the symplectomorphism group.

Applying this idea informally, $\bar{T}_{id_M} = M \times T^2$ admits circle actions in two independent directions (hence a rank 2-lattice many of them); whereas circle action in one direction is broken for $\bar{T}_\phi$. 
This argument fails for

\[ T_\phi = M \times (\mathbb{R} \times S^1 \setminus \mathbb{Z} \times 1)/(x, t, s) \sim (\phi(x), t + 1, s) \]

The circle action is broken on \( T_0 = T^2 \setminus \{\ast\} \).
How to apply flux in this case?

- We may try to partially compactify $T_\phi$
- Hard to characterize uniquely
- Heuristically partial compactifications correspond to deformations of the Fukaya category
- Hence, we wish to apply the idea of flux to $\mathcal{W}(T_\phi)$
- We propose an categorical model for the mapping torus and prove an abstract result instead

**Advantage:** Applies to manifolds $X$ such that $\mathcal{W}(X) \simeq \mathcal{W}(T_\phi)$.

**Work in progress:** Have to relate the abstract categorical mapping tori to $\mathcal{W}(T_\phi)$. 
Mapping torus categories

Let $\mathcal{A}$ be an $A_\infty$ category over $\mathbb{C}$ and $\phi$ be an $A_\infty$-autoequivalence. Further assume

1. $\mathcal{A}$ is smooth, i.e. the diagonal bimodule is perfect
2. $\mathcal{A}$ is proper in each degree and bounded below
3. $\text{HH}^i(\mathcal{A}) = 0$ for $i < 0$ and $\text{HH}^0(\mathcal{A}) \cong \mathbb{C}$

Associated to this data we construct a category $M_\phi$, the mapping torus category satisfying the properties 1-3.
Sketch of the construction

Let \( \tilde{T}_0 \) denote the Tate curve. It is a chain of \( \mathbb{P}^1 \)'s defined by gluing \( \text{Spec}(\mathbb{C}[X_i, Y_{i+1}] / X_i Y_{i+1}) \)

Note the natural right translation automorphism \( \text{tr} \curvearrowright \tilde{T}_0 \) and the \( \mathbb{G}_m \) action. Locally, \( z \in \mathbb{G}_m \) acts by \( X_i \mapsto z^{-1} X_i, Y_{i+1} \mapsto z Y_{i+1} \)
We find a dg category $\mathcal{O}(\tilde{T}_0)_{dg}$ such that

1. $tw^\pi(\mathcal{O}(\tilde{T}_0)_{dg})$ is a dg enhancement for $D^b(Coh_p(\tilde{T}_0))$, bounded derived category of coherent sheaves with a support of finite type

2. $tr = tr_*$ acts strictly on $\mathcal{O}(\tilde{T}_0)_{dg}$

3. The geometric $\mathbb{G}_m$-action above induces a nice action on $\mathcal{O}(\tilde{T}_0)_{dg}$

Moreover, $ob(\mathcal{O}(\tilde{T}_0)_{dg}) = \{\mathcal{O}_{C_i}(-1), \mathcal{O}_{C_i} : i \in \mathbb{Z}\}$.

Consider $\mathcal{O}(\tilde{T}_0)_{dg} \otimes \mathcal{A}$, which carries a $\mathbb{Z}_r$-action generated by $tr \otimes \phi$.

**Definition**

The mapping torus category is defined as $M_\phi := (\mathcal{O}(\tilde{T}_0)_{dg} \otimes \mathcal{A})\# \mathbb{Z}$
Reminder on smash products

Given a dg category $\mathcal{B}$ with a (nice) action of the discrete group $G$, we can construct a category $\mathcal{B}^\# G$ such that

1. $\text{ob}(\mathcal{B}^\# G) = \text{ob}(\mathcal{B})$
2. $(\mathcal{B}^\# G)(b, b') = \bigoplus_{g \in G} \mathcal{B}(g.b, b')$. Let $f \in \mathcal{B}(g.b, b')$ be denoted by $f \otimes g$
3. $(f \otimes g)(f' \otimes g') = fg(f') \otimes gg'$

Morally, if $\mathcal{B}$ has geometric origin this gives a category associated to quotient by $G$.

Remark

The $\mathbb{G}_m$-action on $\mathcal{O}(\tilde{T}_0)_{dg}$ induces a $\mathbb{G}_m$-action on $M_\phi$. 
We are now ready to state the main theorem:

**Main theorem**

Assume further $HH^1(A) = HH^2(A) = 0$. If $M_\phi$ and $M_{1_A}$ are Morita equivalent then $\phi \simeq 1_A$. 
Reminder on Morita equivalences

Given two $A_\infty$-categories $\mathcal{B}_1$ and $\mathcal{B}_2$, we call them Morita equivalent if there is a $\mathcal{B}_1$-$\mathcal{B}_2$-bimodule $E$ and a $\mathcal{B}_2$-$\mathcal{B}_1$-bimodule $E'$ such that

$E \overset{L}{\otimes}_{\mathcal{B}_2} E' \simeq \mathcal{B}_1$ and $E' \overset{L}{\otimes}_{\mathcal{B}_1} E \simeq \mathcal{B}_2$. By Toen’s work they are Morita equivalent if and only if $tw^\pi(\mathcal{B}_1)$ and $tw^\pi(\mathcal{B}_2)$ are $A_\infty$-equivalent.
Given a variety $X$ and automorphism $\phi_0 \curvearrowright X$ construct

$$M_{\phi_0}^{AG} = \tilde{T}_0 \times X/(t, x) \sim (\text{tr}(t), \phi_0(x)) \cong \mathbb{P}^1 \times X/(0, x) \sim (\infty, \phi_0(x))$$

Remark

We expect $D^b(Coh(M_{\phi_0}^{AG})) \cong H^0(tw^\pi(M_{\phi}))$ for $\phi = (\phi_0)_\ast$. 
Before we sketch the proof of the main theorem let us give the basic idea on $M^{AG}_{\phi_0}$. $M^{AG}_{\phi_0}$ is fibered over $\mathcal{J}_0$, the nodal elliptic curve and it has a natural deformation over $Spf(R) = Spf(\mathbb{C}[[q]])$.
Here $\mathcal{T}_R$ denotes the Tate family, a natural smoothing of the nodal elliptic curve. One way to define the deformation $M^{AG,R}_{\phi_0}$ is to use the formal smoothing $\tilde{T}_R$ of $\tilde{T}_0$ locally given by $\text{Spf}(\mathbb{C}[X_i, Y_{i+1}][[q]]/(X_iY_{i+1} - q))$.

Then $M^{AG}_{\phi_0} := \tilde{T}_R \times X/(t, x) \sim (\text{tr}(t), \phi_0(x))$.
Geometric idea

1. Pass to generic fiber $M_{\phi_0}^{AG,K}$ of $M_{\phi_0}^{AG,R}$ to obtain an analytic mapping torus over $K = \mathbb{C}((q))$

2. There is an action of the generic fiber $\mathcal{T}_K$ of $\mathcal{T}_R$ on $M_{1x}^{AG,K} = \mathcal{T}_K \times X$(in a specific direction)

3. This action is broken on $M_{\phi_0}^{AG,K}$ unless $\phi_0 = 1_X$
Notice the same idea can be phrased in terms of $\mathbb{G}^\text{an}_{m,K}$-action on $M^{AG,K}_{\phi_0}$ which restricts to fiberwise action of $\phi_0$ at $t = q$. This is essentially a flow line along a given direction. We will apply a categorical version of this idea, but instead of using generic fibers we will prove results up to $q$-torsion. Instead of flow lines, we will use family of “endo-functors” or bimodules parametrized by a formal scheme whose generic fiber gives $\mathbb{G}^\text{an}_{m,K}$, namely $\tilde{\mathcal{F}}_R$. 
Need a categorical analogue of $M^{AG,R}_{\phi_0}$

Deform $\mathcal{O}(\tilde{T}_0)_{dg}$ to obtain a curved dg category $\mathcal{O}(\tilde{T}_R)_{cdg}$ over $R = \mathbb{C}[[q]]$ with action of $\text{tr}$

Let $M^R_{\phi} := (\mathcal{O}(\tilde{T}_R)_{cdg} \otimes \mathcal{A}) \# \mathbb{Z}$
We construct a family of endo-functors/bimodules of $M^R_{\phi}$ parametrized by $\text{Spf}(\mathbb{C}[u, t][[q]]/(ut - q)) \hookrightarrow \tilde{T}_R$

First define it for $\mathcal{O}(\tilde{T}_R)_{cdg}$ by utilizing a “graph” in $\mathcal{G}_R \subset \tilde{T}_R \times \tilde{T}_R \times \text{Spf}(A_R)$

In local coordinates, $\mathcal{G}_R$ is given by

$$tY_{i+1} = Y'_{i+1}, \quad tX_i' = X_i, \quad Y_{i+1}X_i' = u \text{ or}$$

$$Y_{i+1} = uY_i', \quad X_{i-1}' = uX_i, \quad Y_i'X_i = t$$

This graph naturally extends to $\tilde{T}_R \times \tilde{T}_R \times \tilde{T}_R$ and in the generic fiber we expect the graph of $\mathcal{G}^{an}_{m,K} \times \mathcal{G}^{an}_{m,K} \to \mathcal{G}^{an}_{m,K}$ sending $(z_1, z_2) \mapsto z_1^{-1}z_2$
Imagine the part of $\mathcal{G}_R|_{q=0}$ on $t$-axis as degeneration of the action and the part on the $u$-axis as the degeneration of the inverse action composed with backwards translation.

$\mathcal{G}_R|_{t=1} = \Delta_{\tilde{J}_R}, \mathcal{G}_R|_{u=1} = graph(t\tau^{-1})$
The family of bimodules on $M^R_\phi$

- First define an $A_R$-valued bimodule on $O(\tilde{T}_R)_{cdg}$ by "$(\mathcal{F}, \mathcal{F}') \mapsto \text{hom}_{\tilde{T}_R \times \tilde{T}_R} (q^* \mathcal{F}, p^* \mathcal{F}' \otimes \mathcal{G}_R)"

- Then descent to $M^R_\phi = (O(\tilde{T}_R)_{cdg} \otimes A) \# \mathbb{Z}$

We obtain an $A_R$-valued bimodule $\mathcal{G}_R$; hence, a module over $M^R_\phi \otimes (M^R_\phi)^{op} \otimes A_R$. 
We prove \( \mathcal{G}_R \) is a family of \( M^R_\phi \)-bimodules (parametrized by \( \text{Spf}(A_R) \)) satisfying

1. \( \mathcal{G}_R|_{q=0} \) can be represented by a twisted complex over \( M_\phi \otimes M^{\text{op}}_\phi \otimes \mathbb{C}[u, t]/(ut) \).

2. The restriction \( \mathcal{G}_R|_{t=1} \) is isomorphic to diagonal bimodule of \( M^R_\phi \).

3. \( \mathcal{G}_R \) follows the class \( 1 \otimes \gamma^R_\phi \in HH^1(M^R_\phi \otimes M^{R;\text{op}}_\phi, M^R_\phi \otimes M^{R;\text{op}}_\phi) \) along the direction \( t\partial_t - u\partial_u \).

Here \( \gamma^R_\phi \) is a distinguished class in \( HH^1(M^R_\phi, M^R_\phi) \). We will explain the terms “family” and “follows”. We show the properties 1-3 uniquely characterize the family \( \mathcal{G}_R \) up to \( q \)-torsion.
Briefly families of (bi)modules

Given an $A_\infty$-category $\mathcal{B}$ and a affine variety/formal scheme $S$, we can define a family of (bi)modules parametrized by $S$ to be an $(A_\infty)$-(bi)module $\mathcal{M}$ over $\mathcal{B}$ which carries the structure of a (graded)free $\mathcal{O}(S)$-module such that the $\mathcal{B}$-(bi)module maps are $\mathcal{O}(S)$-linear. Define a morphism of families to be an $A_\infty$ $\mathcal{B}$-(bi)module homomorphism that is $\mathcal{O}(S)$-linear.
We wish to measure the “rate of change” of the family along a derivation $D_S$ on $\mathcal{O}(S)$.

For simplicity consider only families of right modules. Let $\mathcal{M}$ be a family of right modules. Define a pre-connection $\mathcal{D}$ along $D_S$ on $\mathcal{M}$ to be a collection of maps

\begin{align*}
\mathcal{D}^1 & : \mathcal{M}(b_0) \to \mathcal{M}(b_0) \\
\mathcal{D}^2 & : \mathcal{M}(b_1) \otimes \mathcal{B}(b_0, b_1) \to \mathcal{M}(b_0)[-1]
\end{align*}

\ldots

such that $\mathcal{D}^i$ is $\mathcal{O}(S)$-linear for $i \geq 2$ and $\mathcal{D}^1$ satisfies the Leibniz rule with respect to $D_S$, i.e. $\mathcal{D}^1(fs) = f\mathcal{D}^1(s) + D_S(f)s$. 
$\mathcal{D}$ can be thought as an $A_\infty$-pre-module map and its differential, denoted by $\text{def}(\mathcal{D})$ gives a class

$$\text{def}(\mathcal{D}) \in \text{hom}^1_{\mathcal{B}_{\mathcal{O}(S)}^\text{mod}} (\mathcal{M}, \mathcal{M})$$

where $\mathcal{B}_{\mathcal{O}(S)}^\text{mod}$ is the category of families of right $\mathcal{B}$-modules parametrized by $S$. In particular, it is closed and $\mathcal{O}(S)$-linear and the cohomology class $[\text{def}(\mathcal{D})]$ is independent of the choice of pre-connection $\mathcal{D}$. Denote it by $\text{Def}(\mathcal{M})$. 
Let $\gamma \in CC^1(\mathcal{B}, \mathcal{B})$. It induces an endomorphism of degree 1 on every $\mathcal{B}$-module and in particular a cochain

$$\gamma_{\mathcal{M}}^{mod,0} \in \text{hom}^1_{\mathcal{B}^{mod}_{\mathcal{O}(S)}}(\mathcal{M}, \mathcal{M})$$

If $\gamma$ is closed and $[\gamma_{\mathcal{M}}^{mod,0}] = \text{Def}(\mathcal{M})$ we say $\mathcal{M}$ follows $\gamma$. 
Let $\mathcal{O}(S) = A_R := \mathbb{C}[u, t][[q]]/(ut - q)$ and $D_{AR} := t\partial_t - u\partial_u$. This derivation can be seen as the infinitesimal action of $z\partial_z \in Lie(\mathbb{G}_m)$, where $z \in \mathbb{G}_m$ acts by $t \mapsto zt, u \mapsto z^{-1}u$.

Assume there is a (nice) $\mathbb{G}_m$-action on $B$. Then again $z\partial_z \in Lie(\mathbb{G}_m)$ induces a class $(z\partial_z)^\# \in HH^1(B, B)$, the infinitesimal action.

**Lemma**

Assume a family $\mathcal{M}$ carries a (nice) $\mathbb{G}_m$-equivariant structure. Then $\mathcal{M}$ admits a natural pre-connection and follows the class $[(z\partial_z)^\#]$. 
The graph $\mathcal{G}_R \subset \tilde{T}_R \times \tilde{T}_R \times \text{Spf}(A_R)$, which is locally given by

$$tY_{i+1} = Y'_{i+1}, tx'_i = x_i, Y_{i+1}x'_i = u \text{ or }$$

$$Y_{i+1} = uy'_i, x'_{i-1} = ux_i, y'_ix_i = t$$

is $\mathbb{G}_m$-invariant, where $\mathbb{G}_m$ acts by $z : t \mapsto zt, u \mapsto z^{-1}u$ and $z : x'_i \mapsto z^{-1}x'_i, y'_i \mapsto zy'_i$ (i.e. trivially in the first component and as before in the second and third components).

Let $\gamma^R_\phi = (z\partial_z)^\#$:

**Corollary**

$\mathcal{G}_R$ follows the class $1 \otimes \gamma^R_\phi$. 
Proposition

Let $\mathcal{G}_R'$ be another family of bimodules satisfying 1-3. Then, there exists morphisms $f : \mathcal{G}_R \to \mathcal{G}_R'$ and $g : \mathcal{G}_R' \to \mathcal{G}_R$ in the category $H^0((M^R_{\phi})_{bimod})$ - the homotopy category of families of bimodules - such that $f \circ g = q^N 1_{\mathcal{G}_R'}$, $g \circ f = q^N 1_{\mathcal{G}_R}$ for some $N$.

Hence, the family $\mathcal{G}_R$ is characterized by 1-3 up to $q$-torsion.
Proof of the uniqueness

Consider the chain complex $\text{hom}_{(M^R)_{\phi \otimes A_R}^{\text{bimod}}}(G_R, G'_R) = \text{hom}(G_R, G'_R)$. It is a complex of flat $A_R$-modules and its cohomology is finitely generated over $A_R$ in each degree (thanks to Property 1). This complex carries a connection along $D_{A_R}$ in each degree given by

$$\left(\mathcal{D}_{G'_R} \circ (\cdot) - (\cdot) \circ \mathcal{D}_{G_R}\right)$$

Call such a collection of connections a pre-connection on the complex and denote it by $\mathcal{D}$. 

The class of $at(\mathcal{D}) := d \circ \mathcal{D} - \mathcal{D} \circ d$ is given by

$$\text{def}(\mathcal{D}_{\mathcal{R}'_R}) \circ (\cdot) - (\cdot) \circ \text{def}(\mathcal{D}_{\mathcal{R}_R})$$

By Assumption 2 on families, $\text{def}(\mathcal{D}_{\mathcal{R}_R})$, resp. $\text{def}(\mathcal{D}_{\mathcal{R}'_R})$ is cohomologous to $\gamma_{\mathcal{R}_R}^{\text{mod},0}$, resp. $\gamma_{\mathcal{R}'_R}^{\text{mod},0}$ (where $\gamma = 1 \otimes \gamma^R$); hence

$$at(\mathcal{D}) \simeq \gamma_{\mathcal{R}'_R}^{\text{mod},0} \circ (\cdot) - (\cdot) \circ \gamma_{\mathcal{R}_R}^{\text{mod},0}$$

But this is null-homotopic, where the homotopy is given by a natural element $\gamma^{\text{mod},1} : \text{hom}^0(\mathcal{R}_R, \mathcal{R}'_R) \to \text{hom}^0(\mathcal{R}_R, \mathcal{R}'_R)$. 
Let $C^*$ be a complex of $A_R$-modules and endow each $C^i$ with a connection along $D_{A_R}$. Let $\mathcal{D}$ denote this pre-connection. As before, $at(\mathcal{D}) := d(\mathcal{D}) = d \circ \mathcal{D} - \mathcal{D} \circ d$.

**Lemma**

Assume $at(\mathcal{D}) = d(h) = d \circ h - h \circ d$ for $h \in \text{hom}^0(C^*, C^*)$. Then, $h$ can be used to correct $\mathcal{D}$ so that $\mathcal{D}$ becomes a chain map.

In particular, $\text{hom}^1(G_R, G'_R)$ is a complex of $A_R$-modules with connections and the collection of connections form a chain map.

**Corollary**

$\text{Hom}(G_R, G'_R) = H^0(\text{hom}^1(G_R, G'_R))$ is a finitely generated $A_R$-module with a connection.
The special choice \( \gamma^{\text{mod},1} \) of null-homotopy makes sure that compositions such as

\[
\text{Hom}(\mathcal{G}'_R, \mathcal{G}_R) \otimes_{A_R} \text{Hom}(\mathcal{G}_R, \mathcal{G}'_R) \to \text{Hom}(\mathcal{G}_R, \mathcal{G}_R)
\]

are also compatible with the induced connections.
Before proceeding the proof of uniqueness, let us make a remark about $\text{Hom}(G_R, G'_R)|_{t=1}$. As expected, it is isomorphic to $\text{Hom}(G_R|_{t=1}, G'_R|_{t=1})$ but this relies on the existence of connection on the complex $\text{hom}(G_R, G'_R)$.

**Lemma**

$HH^0(M^R_\phi, M^R_\phi) \cong R.$

**Corollary**

$\text{Hom}(G_R, G'_R)|_{t=1} = \text{Hom}(G_R, G'_R)/(t-1)\text{Hom}(G_R, G'_R) \cong R.$

The rest of the proof of uniqueness depends on commutative algebra of modules over $A_R = \mathbb{C}[u, t][[q]]/(ut - q)$. 
Lemma

Let $M$ be a finitely generated $A_R$-module which carries a connection $D_M$ along $D_{A_R}$. Assume $M|_{t=1} = M/(t-1)M$ is $q$-torsion over $A_R/(t-1)A_R = R$. Then $M$ is $q$-torsion.

Lemma

Let $M$ be a finitely generated $A_R$-module which carries a connection $D_M$ along $D_{A_R}$. Then, $M$ is free up to $q$-torsion.

Corollary

$\text{Hom}(\mathcal{G}_R, \mathcal{G'}_R)$ is isomorphic to $A_R = \mathbb{C}[u, t][[q]]/(ut - q)$ up to $q$-torsion.
Consider

$$\text{Hom}(G'_R, G_R) \otimes_{A_R} \text{Hom}(G_R, G'_R) \rightarrow \text{Hom}(G_R, G_R)$$

All the modules involved carry connections and $\text{Hom}(G'_R, G_R)$ etc. are isomorphic to $A_R$ up to $q$-torsion. Up to $q$-torsion it is equivalent to

$$A_R \otimes_{A_R} A_R \rightarrow A_R$$

and thus we can lift $q^N1_{G_R}$ for some $N$. Same in the other direction.
Proof of the main theorem

Assume $M_{\phi}$ and $M_{1,A}$ are Morita equivalent.

Claim

$M^R_\phi$ is Morita equivalent to $\psi^* M^R_{1,A}$ where $\psi_q$ is a transformation of $R$.

This holds since the only deformation of $M_{\phi}$ that is non-trivial in the first order is $M^R_\phi$. For simplicity assume $\psi_q = 1_R$ and $M^R_\phi$ is Morita equivalent to $M^R_{1,A}$. 
Claim

\[ HH^1(M^R_\phi, M^R_\phi) \cong HH^1(M^R_{1A}, M^R_{1A}) \cong R^2 \] and the Morita equivalence can be modified so that the natural isomorphism carries \( \gamma^R_\phi \) to \( \gamma^R_{1A} \).

\[ M^R_{1A} \cong \mathcal{A} \otimes M^R_{1C} \] and \( M_{1C} \) is a model for \( D^b \text{Coh} (\mathcal{T}_0) \cong D^\pi \mathcal{W}(T_0) \) where \( \mathcal{T}_0 \) is the nodal elliptic curve and \( T_0 \) is the punctured torus. Hence, it has sufficient symmetries to modify the Morita equivalence.

Remark

Heuristically, \( HH^1(\mathcal{B}, \mathcal{B}) \) can be thought as the Lie algebra of \( Auteq(tw^\pi(\mathcal{B})) \). In our situation we have a natural copy of \( \mathbb{Z}^2 \) inside \( HH^1(\mathcal{B}, \mathcal{B}) \)- the coroots- and the classes above fall into these discrete subgroups.
The Morita equivalence gives a correspondence between families of bimodules parametrized by $\text{Spf}(A_R)$. Moreover, the family $(\mathcal{G}_R)_{1_A}$ corresponds to still satisfies 1-3. Hence, it is the same as $(\mathcal{G}_R)_\phi$ up to $q$-torsion.

**Remark**

$(\mathcal{G}_R)_{1_A}|_{u=1}$ is isomorphic to diagonal and $(\mathcal{G}_R)_\phi|_{u=1}$ is isomorphic to “fiberwise $\phi$”.

Fiberwise $\phi$ is an auto-equivalence of $M^R_\phi$ that is given by the descent of $\text{tr}^{-1} \otimes 1_A$ or $1_{\mathcal{O}(\tilde{\mathcal{T}}_R)_{cdg}} \otimes \phi$ on $\mathcal{O}(\tilde{\mathcal{T}}_R)_{cdg}$. This implies fiberwise $\phi$ is the same as $1_A$ up to $q$-torsion.
Pick a smooth $R$-point $p$ on the deformation of nodal curve. Any $a \in ob(A)$ we have an unobstructed object "$\mathcal{O}_p \otimes a"$ over $M^R_{\phi}$ and a subcategory $\{\mathcal{O}_p\} \otimes A$. Fiberwise $\phi$ induces $1 \otimes \phi$ on $\{\mathcal{O}_p\} \otimes A$ and it is the same as the diagonal bimodule up to $q$-torsion. Hence, after inverting $q$, they are the same and this easily implies $\phi \simeq 1_A$. 