On the Structure of Overgroup Lattices in Exceptional Groups of Type $G_2(q)$

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Abstract: We study overgroup lattices in groups of type $G_2$ over finite fields and reduce Shareshian’s conjecture on $D\Delta$-lattices in the $G_2$ case to a smaller question about the number of maximal elements that can be conjugate in such a lattice. Here we establish that, modulo this smaller question, no sufficiently large $D\Delta$-lattice is isomorphic to the lattice of overgroups $O_G(H)$ in $G = G_2(q)$ for any prime power $q$ and any $H \leq G$. This has application to the question of Palfy and Pudlak as to whether each finite lattice is isomorphic to some overgroup lattice inside some finite group.

1 Introduction

Recall that a lattice is a partially ordered set in which each element has a greatest lower bound and a least upper bound.

In 1980, Palfy and Pudlak [PP] proved that the following two statements are equivalent:

(1) Every finite lattice is isomorphic to an interval in the lattice of subgroups of some finite group.

(2) Every finite lattice is isomorphic to the lattice of congruences of some finite algebra.

This leads one to ask the following question:

The Palfy-Pudlak Question: Is each finite lattice isomorphic to a lattice $O_G(H)$ of overgroups of some subgroup $H$ in a finite group $G$?

There is strong evidence to believe that the answer to Palfy-Pudlak Question is no. While it would suffice to exhibit a single counterexample, John Shareshian has conjectured that an entire class of lattices - which he calls $D\Delta$-lattices - are not isomorphic to overgroup lattices in any finite groups.
2 Preliminaries

We will use the following terminology and notation when discussing such lattices:

Let $\Lambda$ be a finite lattice with greatest element $\infty$ and least element 0, and let $\Lambda'$ be the associated partially ordered set (or poset) $\Lambda - \{0, \infty\}$. If we view this poset as a graph in which two vertices (elements) are adjacent iff they are comparable in the poset $\Lambda'$, then we can borrow the definition of connectedness from graph theory and apply it to our lattice. Specifically, $\Lambda$ is disconnected iff the poset $\Lambda'$ is disconnected as a graph.

For a positive integer $m$, let $\Delta(m)$ denote the poset of all subsets of a set with $m$ elements partially ordered by set inclusion. If a lattice $\Lambda$ is disconnected and $\Lambda'$ has connected components $\Lambda_i$, such that $\Lambda_i \cong \Delta(m_i)'$ for some integers $m_i \geq 3$, $1 \leq i \leq r$, we say $\Lambda$ is a $D\Delta$-lattice. Write $D\Delta(m_1, \ldots, m_r)$ for this lattice. Let $D\Delta[C_2, C_3]$ be the infinite class of $D\Delta(m_1, \ldots, m_r)$ lattices such that $r \geq C_2$ and $m_i \geq C_3$ for each $1 \leq i \leq r$.

In [A1] and [A3] Aschbacher proved that Shareshian’s conjecture can be reduced to two smaller problems, one of which is the primary motivation for this paper:

**Aschbacher’s Reduction** (Part 1 of 2): *If $G$ is an almost simple finite group and $H \leq G$, then $O_G(H)$ is not isomorphic to a $D\Delta$-lattice.*

Now that the finite simple groups have been classified, we know that any finite simple group is isomorphic to either a cyclic group of prime order, an alternating group, a simple group of Lie type, or one of 26 sporadic groups. Aschbacher and Shareshian [AS] have taken care of both parts of the reduction in the case where $G$ is either an alternating group or a symmetric group. If we only wish to show that all but finitely many $D\Delta$-lattices are not isomorphic to overgroup lattices, we can disregard the 26 sporadic groups, as a finite set of finite groups can only have finitely many subgroups and thus finitely many overgroup lattices for those subgroups. All that remain are the groups of Lie type.

This paper considers the case when $G = G_2(q)$ is a group of type $G_2$ over a finite field with $q$ elements. We make use of a theorem of Aschbacher [A2] which completely describes the maximal subgroups of $G_2(q)$, and derive some results about possible occurrences of $\Delta$ and $D\Delta$-lattices in $G_2(q)$. Using these results and one assumption, we obtain our main theorem:

**Theorem 1** Assume that for a prime power $q$, $G_2(q)$ satisfies the conditions of Hypothesis 1 as described in section 3. Then there exist constants $C_2$ and $C_3$ such that for any $H \leq G \cong G_2(q)$, we have that $O_G(H) \notin D\Delta[C_2, C_3]$.  

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The weakness of this theorem is that Hypothesis 1 may be difficult to verify. However, the reduction is still interesting since if Hypothesis 1 can be verified in general, our approach for $G_2(q)$ should be able to be extended to all of the exceptional groups. There are weaker versions of this assumption that should be substantially easier to prove, and the intermediate results shown here can reduce the work required in such an alternative approach.

To begin, we need a few preliminary results on $\Delta$ and $D\Delta$-lattices.

### 2.1 $\Delta$ and $D\Delta$-Lattices

Let $G$ be a finite group and $H$ be a subgroup of $G$. Write $O_G(H)$ for the poset of subgroups of $G$ containing $H$ partially ordered by inclusion. This poset is the lattice of overgroups of $H$ in $G$. Recall the definition of $D\Delta(m_1, \ldots, m_r)$ from the introduction.

**Lemma 2.1** Suppose $O_G(H)$ is isomorphic to $D\Delta(m_1, \ldots, m_r)$. Then for any $K \in O_G(H) \setminus \{G\}$, we have $O_K(H) \cong \Delta(m')$ for some integer $m'$.

Proof: Let $M$ be a maximal member of the connected component $C_k$ in $O_G(H)$. As $\Delta(m_k)$ is the lattice of subsets of an $m_k$-set, $M$ corresponds to an $(m_k - 1)$-set, and the vertices under $M$ correspond to all proper subsets of that $(m_k - 1)$-set, i.e. the lattice $\Delta(m_k - 1)$. Inductively, the claim holds for any $K \in O_G(H) \setminus \{G\}$. □

**Lemma 2.2** Suppose $O_G(H)$ is isomorphic to a $D\Delta$-lattice or a $\Delta$-lattice and let $K$ be a subgroup of $G$ properly containing $H$. Then $O_G(K) \cong \Delta(m')$ for some integer $m'$.

Proof: Suppose $K$ belongs to the connected component $C$ of $O_G(H)$, and so $C \cong \Delta(m)$ for some $m$. Thus, there is a poset isomorphism $\pi$ mapping elements of $C$ to subsets of a set $S$ with $|S| = m$. Clearly, $\pi$ restricts to a poset isomorphism of $O_G(K)$ with $S \setminus \pi(K)$, so $O_G(K) \cong \Delta(m')$, where $m' = |S \setminus \pi(K)|$. □

The following lemma motivates the first definition in the next section:

**Lemma 2.3** (Shareshian) Suppose $O_G(H)$ is a $D\Delta$-lattice. Then for each $K \in O_G(H)$, we have $K = N_G(K)$. 
3 \((C_2, C_3)\)-setups

Given a finite group \(G\), define

\[
n(G) = \max\{m : O_G(K) \cong \Delta(m) \text{ for some subgroup } K \leq G \text{ such that } K = N_G(K)\}
\]

Let \(C_2\) and \(C_3\) be positive integers. Let \(\mathcal{G} = (G(q) : q \in I)\) be a family of finite groups indexed by a parameter \(q\). Let \(\mathcal{M}_i\) be the family \((\mathcal{M}_i(q) : q \in I)\), where \(\mathcal{M}(q)\) is the set of maximal subgroups of \(G(q)\), and \(\mathcal{M}(q) = \mathcal{M}_1(q) \cup \mathcal{M}_2(q) \cup \mathcal{M}_3(q)\) for \(q \in I\).

We say the 4-tuple \((\mathcal{G}, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)\) is a \((C_2, C_3)\)-setup if for each \(q \in I\) we have:

1. the number of \(G(q)\) conjugacy classes in \(\mathcal{M}_2(q)\) is less than \(C_2\), and
2. \(n(M) < C_3 - 1\) for all \(M \in \mathcal{M}_3\).

**Hypothesis 1** A \((C_2, C_3)\)-setup \((\mathcal{G}, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)\) will be said to satisfy Hypothesis 1 if for each \(q \in I\), \(M \in \mathcal{M}_1(q) \cup \mathcal{M}_2(q)\), and \(H \leq M\) such that \(O_{G(q)}(H) \cong \Delta(m)\) for some \(m \geq C_3 - 1\), we have that \(M\) is the unique conjugate of itself that contains \(H\).

Let \(D\Delta[C_2, C_3]\) be the infinite class of \(D\Delta(m_1, \ldots, m_r)\) lattices such that \(r \geq C_2\) and \(m_i \geq C_3\) for each \(1 \leq i \leq r\).

**Lemma 3.1** Suppose \((\mathcal{G}, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)\) is a \((C_2, C_3)\)-setup satisfying Hypothesis 1, and that for some \(q \in I\) and \(H \leq G(q)\) we have that \(O_{G(q)}(H) \in D\Delta[C_2, C_3]\). Then

1. \(O_{G(q)}(H) \cap \mathcal{M}(q) \subseteq \mathcal{M}_1(q) \cup \mathcal{M}_2(q)\),
2. for each \(M \in \mathcal{M}(q)\), we have \(|M^{G(q)} \cap O_{G(q)}(H)| \leq 1\), and
3. there exists a connected component \(\mathcal{C}\) of \(O_{G(q)}(H)\) such that \(\mathcal{M}(q) \cap \mathcal{C} \subseteq \mathcal{M}_1(q)\).

Proof: As \(O_{G(q)}(H) \in D\Delta[C_2, C_3]\), we know \(m_i \geq C_3\) for each \(i\). Suppose there exists an \(M \in O_{G(q)}(H) \cap \mathcal{M}_3(q)\). As \(M\) is maximal, \(N_G(M) = M\). By 2.1, we know \(O_M(H) \cong \Delta(m'_i)\), for some integer \(m'_i \geq C_3 - 1\). However, as \(O_{G(q)}(H)\) is a \((C_2, C_3)\)-setup, \(m'_i \leq n(M) < C_3 - 1\), a contradiction. So, \(O_{G(q)}(H) \cap \mathcal{M}_3(q) = \emptyset\) and (1) holds.

Now, as (1) implies that any maximal element of \(O_{G(q)}(H)\) must belong to \(\mathcal{M}_1(q) \cup \mathcal{M}_2(q)\), (2) follows from the fact that \((\mathcal{G}, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)\) satisfies Hypothesis 1.

By (2), we know that \(|O_{G(q)}(H) \cap \mathcal{M}_2(q)|\) is bounded above by the number of conjugacy classes in \(\mathcal{M}_2(q)\), which must be less than \(C_2\) by the definition of a \((C_2, C_3)\)-setup. However, \(O_{G(q)}(H)\) has at least \(C_2\) connected components. Thus, there exists at least one connected component \(\mathcal{C}\) such
that \( M_2(q) \cap C = \emptyset \), and so (3) follows from (1).

4 Subgroups in \( G_2(q) \)

Let \( I \) denote the set of prime powers. For each \( q \in I \), let \( G_2(q) \) be the exceptional group of type \( G_2 \) over the finite field with \( q \) elements. By studying the groups of in the family \( (G_2(q) : q \in I) \) as the groups of automorphisms of two polynomial forms on a 7-dimensional vector space, Aschbacher [A2] has described the maximal subgroups of \( G_2(q) \). I have displayed and collected these groups into classes \( C_1, C_2, \) and \( C_3 \) in the following theorem (see [A2] for full explanation of the notation used).

Theorem on Maximal Subgroups (Aschbacher) Let \( q \) be a power of a prime \( p \). Then each maximal subgroup of \( G_2(q) \) is conjugate to exactly one of the following:

Class \( C_1 \):

(1) \( q \) is not a prime, and \( M \cong G_2(K) \) for some maximal subfield \( K \) of \( F \). If \( q = p^n \), where \( n = p_1^{e_1} \cdots p_r^{e_r} \), then \( |K| = q^{n/p} = p^{n/p_i} \) for some \( 1 \leq i \leq r \).

Class \( C_2 \):

(2) One of two classes of maximal parabolics of \( G_2(q) \).

(3) A stabilizer of \( U_4 \), as described in 2.7 of [A2] as the special orthogonal group \( SO(U_4, B) \), which is the central product of two copies of \( SL_2(q) \), extended by an element inducing an outer automorphism in \( PGL_2(q) \) on both copies when \( q \) is odd.

(4) The normalizer of one of two classes of \( B \)-nondegenerate hyperplanes of \( V \):

(i) The hyperbolic hyperplanes are conjugate to \( V_6 \) with normalizer \( K_0 = K\langle r \rangle \) where \( K = SL(V_3) \cong SL_3(q) \) and \( r \) is an involution inducing the transpose-inverse map on \( K \).

(ii) The other class of hyperplanes has representative \( U \) from section 6 of [A2]. As on page 217 in [A2], let \( k \) be a quadratic extension of \( F \). Then the \( F \)-vector space structure on \( U \) extends to a \( k \)-vector space structure \( U^k \) and \( N_G(U) \) preserves a unitary form \( \alpha \) on \( U^k \) such that \( N_G(U) \) is \( SU(V^k, \alpha) \cong SU_3(q) \) extended by a graph-field automorphism.

(5) \( p = 3 \), and \( M \) is conjugate via an outer automorphism to a subgroup of type (4).
(6) \( q = 3^m \), \( m \) odd, and \( M \cong 2G_2(q) \).

(7) \( p > 5 \) and \( M \cong PGL_2(F_q) \) where \( V \) is the space of homogenous polynomials of degree 6 spanned by the vectors \( \{ x^6, x^5y, x^4y^2, x^3y^3, x^2y^4, xy^5, y^6 \} \) as an \( FM \)-module.

Class \( C_3 \):

(8) \( q = p > 2 \) and \( M \cong L_3(2)/E_8 \).

(9) \( p \neq 13, 13 \) has a square root in \( F_q \) and \( M \cong L_2(13) \)

(10) \( q = p > 3 \) and \( M \cong G_2(2) \).

(11) \( p > 3 \), the polynomial \( X^3 - 3X + 1 \) splits in \( F_q \) and \( M \cong L_2(8) \).

(12) \( q = 11 \) and \( M \cong J_1 \).

(13) \( q = 4 \) and \( M \cong H.J. \)

Further, \( G_2(q) \) acts transitively by conjugation on each of the above types of its maximal subgroups.

Lang’s Theorem  Let \( G \leq GL_n(k) \) be a connected linear algebraic group defined over an algebraic closure \( k \) of a finite field with characteristic \( p \). Let \( \sigma \) be an endomorphism which raises entries of the members of \( G \) to the \( q \)th power, where \( q \) is a power of \( p \). Then the Lang map defined by \( g \mapsto g^{-1}\sigma(g) \) is surjective.

Proof: See [M] for a quick proof of this well known theorem.

Let \( G = G_2(q) \) where \( q = p^n \) for some prime \( p \), and let \( d \) be a divisor of \( n \). Let \( \lambda \) be a Frobenius automorphism on \( G \) defined by the map \( a \mapsto a^p \) for \( a \in F_q \). Let \( G^* \) be the semidirect product \( G(\lambda) \). Let \( \sigma \) be the automorphism \( \lambda^d \), and so \( |\sigma| = n/d \).

Corollary 4.1 \( G \) is transitive on the elements of the coset \( \sigma G \subseteq G^* \) which have order \( |\sigma| \).

Proof: Let \( \bar{F}_q \) be the algebraic closure of \( F_q \) and \( \bar{G} \) be the algebraic group \( G_2(\bar{F}_q) \). Then \( \sigma \) is the
restriction to $G$ of the Frobenius map on $\bar{G}$ defined by $a \mapsto a^{p^d}$ for $a \in F_p$, and we also write $\sigma$ for this automorphism of $G$. As the Lang map which sends $g \in G$ to $g^{-1}\sigma(g)$ is surjective, we know all elements in the coset $\sigma G \subset \bar{G}\langle \sigma \rangle$ are $G$-conjugate to $\sigma$. Let $\tau$ be the map defined by $a \mapsto a^q$, and thus $\bar{G} = G_q = G$. Then $\sigma : a \mapsto a^{p^d}$ is an automorphism of order $n/d$ on $G$ with $\sigma^{n/d} = \tau$. As $\sigma G \subset \sigma G$, if $\alpha \in \sigma G$, then $\alpha^{q^2} = \sigma$ for some $g \in G$. Suppose $\alpha$ has the same order $n/d$ as $\sigma$. Then $\tau = \sigma^{n/d} = (\alpha^q)^{n/d} = (\alpha^{n/d})^q$. As $|\alpha| = n/d$, we have $\alpha^{n/d} \in G_{G(\alpha)}(G)$. This centralizer can be written as the semidirect product $\langle \tau \rangle Z(\bar{G})$, but as $\bar{G}$ is a simple group, $Z(\bar{G}) = 1$, so $\alpha^{n/d} \in C_{G(\alpha)}(G) = \langle \tau \rangle$. But also $\alpha \in \sigma G$ so $\alpha^{n/d} \in \sigma^{n/d}G = \tau G$. Thus $\alpha^{n/d} = \tau = \tau^{q^2}$, and so $g \in \bar{G} = G$. So, $\alpha$ and $\sigma$ are indeed $G$-conjugate as desired.

Lemma 4.2 Let $H \leq G$ with $H \cong G_2(p^d)$. Then $H = G_\alpha$ for some $G$-conjugate $\alpha$ of $\sigma$.

Proof: Let $k = F_{p^d}$. Then, by Theorem 7 in [A2], $H$ stabilizes a $k$-structure (as defined on p. 197 of A2) and $G$ is transitive on $k$-structures. Thus, as $G_\sigma$ also stabilizes a $k$-structure, $H$ and $G_\sigma$ are conjugate - i.e. $gG_\sigma g^{-1} = H$. Thus, elements of $H$ are stabilized by $g\sigma g^{-1} = \alpha$, so $H = G_\alpha$ where $\alpha$ is a $G$-conjugate of $\sigma$.

Lemma 4.3 Let $d \mid c \mid n$, and suppose $H' \leq G$ with $H' \cong G_2(p^d)$. Then we may choose $\lambda$ so that $H' = G_\lambda$, and then $G_{\lambda c}$ is the unique overgroup of $H'$ in $G$ isomorphic to $G_2(p^c)$.

Proof: Suppose $H' \leq K \cong G_2(p^c)$. By 4.2 we may choose $\lambda$ so that $H' = G_\lambda$, where $\sigma = \lambda^d$ and $\lambda$ is a Frobenius endomorphism on $G$. We know $H' \leq H = G_{\lambda c} \cong G_2(p^c)$. By 4.2, we know $K = G_\gamma$ for some $\gamma$ which is $G$-conjugate to $\lambda^c$. As $H' = G_\gamma \leq K$, $\gamma$ centralizes $G_\sigma$ and we also have $\langle \sigma \rangle = C_{G_\gamma}(G_\sigma)$. Thus, as $\gamma \in C_{G_\gamma}(G_\sigma) = \langle \sigma \rangle$, we know $\gamma = \sigma^{c/d} = \lambda^c$. Thus, $K = H$ is unique.

Lemma 4.4 Let $H \leq G = G_2(q)$ with $H \cong G_2(p^d)$. Then $H$ is not contained in any of the groups in the class $C_2$ or the class $C_3$.

Proof: As shown in [A2], $H \cong G_2(p^d)$ acts irreducibly on $(V, f, B)$. Thus, it cannot be a subgroup of a group which acts reducibly. The groups of type (2), (3) and (4) are reducible, as they stabilize some proper subspaces of $V$ (see [A2]).

Now, as any $p$-subgroup is contained in a Sylow $p$-subgroup, if $H$ is a subgroup of some group $M$ in $C_2$, then its Sylow $p$-subgroups are contained in the Sylow $p$-subgroups of $M$. The Sylow $p$-subgroups of $PGL_2(p^k)$ are abelian, while those of $H = G_2(p^d)$ are not, so $H \not\leq M$, for $M$ of type (7). Now, if $p = 3$, we still have to address types (5) and (6). Inside the groups in (5), Sylow 3-subgroups have nilpotence class 2, but Sylow 3-subgroups in $H$ have nilpotence class greater than 2. In (6), Sylow 3-subgroups have the property that they either equal their conjugates or intersect trivially with them (i.e. they are TI-subgroups). Those in $H = G_2(3^d)$ do not have that property. So, $H$ is not contained in a maximal subgroup in the class $C_2$. 7
We know $|H| \geq |G_2(2)| = 12096$. So, $H$ cannot be a subgroup of a group of type (8), (9), or (11), as these groups are too small. If $q$ is prime, $G$ has no proper subgroups of the form $G_2(p^d)$, so we can discard the cases (10) and (12). Now, if $H$ were properly contained in a maximal subgroup $M \cong HJ$, where $HJ$ is the Hall-Janko group, it would be contained in a subgroup $M' < M$ which is maximal in $M$. However, according to the ATLAS of Finite Groups, the largest maximal subgroup of $HJ$ has order 6048. Thus, $H$ is not contained in a group in class $\mathcal{C}_3$.

**Lemma 4.5** Let $H \leq G = G_2(q)$ with $H \cong G_2(p^d)$. Then if $K \in O_G(H)$, we have $K \cong G_2(p^{ad})$, where $a$ is a divisor of $n/d$.

Proof: The claim is trivial if $H = G$ so we may assume $H < G$. We know $H$ is contained in some maximal subgroup $M$ of $G$. By 4.4, $M$ must be $G_2(q^{1/k})$ for some prime $k$ dividing $n$. By induction on $n$, $d$ divides $n/k$ and hence also $n$. Thus the result holds if $K = G$, so we may assume $K$ is proper in $G$ and then choose $K \leq M$. By induction on $n$, $K \cong G_2(p^{ad})$ for some divisor $a$ of $n/kd$, and so the claim holds.

For a positive integer $m$, let $\delta(m)$ denote the lattice of positive divisors of $m$, partially ordered by the division relation.

**Lemma 4.6** Let $H \leq G = G_2(q)$ with $H \cong G_2(p^d)$. Then $O_G(H) \cong \delta(n/d)$.

Proof: By 4.5, we know the members of $O_G(H)$ are those of the form $G_2(p^a)$, where $a$ is a divisor of $n/d$. Then by 4.3, we may take $H = G_\sigma$ and the map $k \mapsto G_{\sigma^k}$ is a bijection of $\delta(n/d)$ with $O_G(H)$. If $a$ divides $b$ then $G_{\sigma^a} \leq G_{\sigma^b}$, while if $G_{\sigma^a} \leq G_{\sigma^b}$ then $a$ divides $b$ by 4.5. Thus the bijection and its inverse preserve the partial order and so $O_G(H) \cong \delta(n/d)$ as posets.

**Corollary 4.7** Let $H \leq G = G_2(q)$ with $H \cong G_2(p^d)$. Then $O_G(H) \cong \Delta(m)$ if and only if $n/d$ is squarefree with $m$ distinct prime factors.

Proof: By 4.6, $O_G(H) \cong \delta(n/d)$. If $n/d$ is squarefree with $m$ distinct prime factors, then the divisors of $n/d$ clearly correspond to subsets of those $m$ primes, and so $\delta(n/d) \cong \Delta(m)$.

Now suppose for some prime $a$ that $a^2$ divides $n/d$. Then the minimal elements of $\delta(n/d)$ are all the prime divisors of $n/d$, including $a$. Any element just above $a$ must contain at least two minimal elements if the lattice is to be a $\Delta$-lattice. However, $a^2$ contains only the element $a$.

**Corollary 4.8** Let $H \leq G = G_2(q)$ with $H \cong G_2(p^d)$. If $n/d$ is not squarefree, then $O_G(H)$ is not a $D\Delta$-lattice.
The Proof of Theorem 1

Proof: Suppose $O_G(H)$ is a $D\Delta$-lattice. Then $O_M(H) \cong \Delta(m)$ for each maximal $M \in O_G(H)$ and some integer $m$. By 4.7, $M \cong G_2(p^m)$ with $d$ dividing $n_M$ and $O_M(H) \cong \delta(n_M/d)$. But if $n/d$ is not squarefree and not equal to $a^2$ for some prime $a$, at least one of the $n_M/d$ is not squarefree, so $O_M(H)$ is not a $\Delta$-lattice by 4.7. If $n/d = a^2$, the lattice is a chain. So, $O_G(H)$ is not a $D\Delta$-lattice.

Lemma 4.9 Let $M_1, M_2$ be maximal subgroups of $G = G_2(p^n)$ with $M_i \cong G_2(p^{n/a_i})$ for some distinct primes $a_1, a_2$ dividing $n$, and let $I = M_1 \cap M_2$. If $O_G(I) \cong \Delta(2)$, then $I \cong G_2(p^{n/\delta_{a_1}a_2})$.

Proof: By 4.2, we know that $M_i = G_{\alpha_i}$ where $\alpha_i$ is a field automorphism of $G$ of order $a_i$. Since $\alpha_i$ centralizes $M_i$, it centralizes $I$, and so $I = \alpha_i(I) \leq \alpha_i(M_2)$. As $\alpha_i$ is an automorphism and $O_G(I) \cong \Delta(2)$, it must be the case that $\alpha_1(M_2) = M_2$, and so $\alpha_1$ restricts to an automorphism on $M_2$ of order $a_1$. By 4.1, we know $\alpha$ is a conjugate under $M_2$ of an element $\alpha_i$ of $\langle \rho \rangle$, where $\rho$ is a field automorphism of $G$ of order $n$ such that $\alpha_2 \in \langle \rho \rangle$. Thus as $\alpha_1$ and $\alpha_2$ are distinct primes, $|\alpha_i^2| = a_1a_2$, so $I^2 = C_{M_2}(\alpha_i) = C_{M_2}(\alpha_i^2) \cong G_2(q^{1/\alpha_1\alpha_2})$, and hence the centralizer of $\alpha_1$ in $M_2$ is isomorphic to $G_2(q^{1/\alpha_1\alpha_2})$. 

5 The Proof of Theorem 1

Let $G = (G_2(q) : q \in I)$ and let $M_i = \{ M \in \mathcal{M}(q) : M \in C_i \}$, for $1 \leq i \leq 3$. Set $C_2 = 9$ and $C_3 = 2 + \max\{ n(M) : M \in C_3 \}$. We call this tuple $(G, M_1, M_2, M_3)$ the $G_2$-setup. By consulting the list in the Theorem on Maximal Subgroups, we see that the $G_2$-setup is indeed a $(C_2, C_3)$-setup.

Theorem 1 Assume the $G_2$-setup satisfies Hypothesis 1. Let $H \leq G = G_2(q)$. Then $O_G(H) \not\in D\Delta[C_2, C_3]$.

Proof: First suppose $q = p$, and $O_{G_2(p)}(H) \in D\Delta[C_2, C_3]$. As $p$ has no proper prime divisors, $M_1(p) = \emptyset$. But according to Lemma 3.1, there exists a connected component of $O_{G_2(p)}(H)$ whose maximal subgroups belong to $M_1(p)$. This contradiction shows $O_{G_2(p)}(H) \not\in D\Delta[C_2, C_3]$.

Now suppose $q$ is not prime and $O_{G_2(q)}(H) \in D\Delta[C_2, C_3]$. We know there exists a connected component $C$ of $O_G(H)$ such that $\{ M_i : M_i \in \mathcal{M}(q) \cap C \} \subseteq M_1(q)$. Because $C$ is a $\Delta$-lattice, the next "row" consists of intersections of pairs of $M_i$ that are maximal in each. Thus, these intersections are of the form described in 4.9. Thus, $O_M(H)$ is a $\Delta$-lattice whose maximal elements are all of the form $G_2(p^d)$. Inductively, $O_G(H)$ consists only of subgroups of the form $G_2(p^d)$, and so $H \cong G_2(p^d)$ for some $d|n$. But then, by 4.7 and 4.8, $O_G(H)$ is not a $D\Delta$-lattice, contradicting $O_G(H) \in D\Delta[C_2, C_3]$. 

9
6 References


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