**Problem I**

As the hint suggests, we will use separation of variables and first consider the homogeneous case. Suppose that we can write \( u(x, t) = c(t) v(x) \). The homogeneous equation gives us

\[
c'(t)v(x) - c(t)v''(x) = 0 \implies \frac{c'(t)}{c(t)} = \frac{v''(x)}{v(x)}.
\]

As in lecture, we get that \( c'(t) = \lambda c(t) \) and \( v''(x) = \lambda v(x) \). Solving the first equation gives \( c(t) = Ae^{\lambda t} \) for some constant \( A \). We consider three cases for \( v(x) \):

- **\( \lambda = 0 \):** Then \( v(x) = Bx + C \) for some constants \( B \) and \( C \). The Neumann data tells us that \( B = 0 \) so \( v(x) \) is a constant in this case. Then the initial data implies that \( u(x, t) = 1 \).

- **\( \lambda > 0 \):** In this case \( v(x) = Be^{\sqrt{\lambda}x} + Ce^{-\sqrt{\lambda}x} \). Taking the first derivative and evaluating at \( x = 0 \) tells us \( B = C \) and evaluating at \( x = L \) tells us \( B = C = 0 \). Thus, \( v(x) = 0 \) in this case.

- **\( \lambda < 0 \):** We have \( v(x) = B \sin(\sqrt{\lambda}|x|) + C \cos(\sqrt{\lambda}|x|) \). Taking the first derivative and evaluating at \( 0 \) tells us \( B = 0 \). Evaluating the first derivative at \( x = L \) tells us \( \sqrt{\lambda} \sin(\sqrt{\lambda}L) = 0 \). The only way this can happen is if

\[ \lambda L^2 = -\pi^2 m^2 \implies \lambda = -\frac{\pi^2 m^2}{L^2}, \quad m \in \mathbb{Z}^+ \]

Therefore, our candidate solution is (the constants are absorbed by \( c(t) \))

\[ u(x, t) = \sum_{m=0}^{\infty} c(t) \cos \left( \frac{m\pi}{L} x \right). \]

The PDE gives us (assuming the sum uniformly converges)

\[
\sum_{m=0}^{\infty} \left( c'(t) + \frac{m^2\pi^2}{L^2} c(t) \right) \cos \left( \frac{m\pi}{L} x \right) = tx. \tag{1}
\]

We now make the observation that \( \{ \cos \left( \frac{m\pi}{L} x \right), m \geq 0 \} \) is an orthogonal set of functions. Namely using integration by parts, we can verify that

\[
\int_{0}^{L} \cos \left( \frac{m\pi}{L} x \right) \cos \left( \frac{n\pi}{L} x \right) \ dx = \begin{cases} 0 & m \neq n. \\ \frac{L}{2} & m = n. \end{cases}
\]

Thus, in the case of \( n \neq 0 \), multiplying both sides of Eq.\((1)\) by \( \cos \left( \frac{n\pi}{L} x \right) \) and integrating from 0 to \( L \) gives us

\[
c'(t) + \frac{n^2\pi^2}{L^2} c(t) = \frac{2}{L} \int_{0}^{L} tx \cos \left( \frac{n\pi}{L} x \right) \ dx.
\]

A quick substitution \( x' \rightarrow \frac{n\pi}{L} x \) tells us that

\[
c'(t) + \frac{n^2\pi^2}{L^2} c(t) = \frac{2tL}{n^2\pi^2} (\cos(n) - 1).
\]
Note that the above expression is nonzero only if \( n \) is odd, in which case it equals \( -\frac{4tL}{n^2 \pi^2} \). For the \( n \) even case we get \( c'(t) = \frac{t L^2}{2} \implies c(t) = \frac{t^2 L^2}{4} \). We must now solve for \( c(t) \) in all of the other cases as well. We wish to solve

\[
c'(t) + \frac{n^2 \pi^2}{L^2} c(t) = -\frac{4tL}{n^2 \pi^2}, \quad n \text{ odd.}
\]

We use integrating factors. Let \( \mu(t) \) be such that \( \mu'(t) = \frac{n^2 \pi^2}{L^2} \mu(t) \). Hence, the LHS is just \( (\mu(t)c(t))' \) so we get that

\[
c(t) = \frac{\int -\frac{4tL}{n^2 \pi^2} \mu(t) \, dt + C}{\mu(t)}
\]

where

\[
\mu(t) = e^{\frac{n^2 \pi^2}{L^2} t}
\]

and \( C \) is some constant. Integrating by parts gives us

\[
c(t) = -\frac{4L^3}{\pi^4 n^4} t + \frac{4L^5}{n^6 \pi^6} + Ce^{-\frac{n^2 \pi^2}{L^2} t}.
\]

We can explicitly solve for \( C \) using the Cauchy data:

\[
u(x, 0) = 1 \implies C = -\frac{4L^5}{n^6 \pi^6}.
\]

Putting everything together gives us

\[
u(x, t) = 1 + \frac{t^2 L^2}{4} + \sum_{n | n \text{ odd}} \left(-\frac{4L^3}{\pi^4 n^4} t + \frac{4L^5}{n^6 \pi^6} - \frac{4L^5}{n^6 \pi^6} e^{-\frac{n^2 \pi^2}{L^2} t} \right) \cos \left( \frac{n \pi}{L} x \right).
\]

**Problem II**

We again use separation of variables. Let \( u(t, x) = c(t)v(x) \) which tells us

\[
c'(t)v(x) - Dc(t)v''(x) = 0 \implies \frac{c'(t)}{Dc(t)} = \frac{v''(x)}{v(x)} = \lambda \implies c(t) = Ae^{\lambda t}
\]

where \( A \) is constant. We again have three cases:

- \( \lambda = 0 \): In this case, \( v(x) = Bx + C \) for constants \( B, C \). Then \( u_x(0, t) = 0 \implies B = 0 \). The last equation then tells us \( u(x, t) = C \cdot c(t) = U \) so we have found a solution for the inhomogeneous mixed condition.

- \( \lambda > 0 \): In this case \( v(x) = Be^{\sqrt{\lambda} x} + Ce^{-\sqrt{\lambda} x} \). Using \( u_x(0, t) = 0 \) implies that \( B = C \). We use the homogeneous equation \( u_x(L, t) + u(L, t) = 0 \) since we already know a particular solution from above. We can temporariliy treat \( c(t) \) as a constant so we have

\[
u_x(L, t) + u(L, t) = 0 \implies \sqrt{\lambda} \frac{e^{\sqrt{\lambda} L} - e^{-\sqrt{\lambda} L}}{e^{\sqrt{\lambda} L} + e^{-\sqrt{\lambda} L}} + 1 = 0.
\]
Note that the LHS of the above equation is just the hyperbolic tangent function so we have
\[ \sqrt{\lambda} \tanh(\sqrt{\lambda} L) + 1 = 0. \]

If we make the substitution \( x' = \sqrt{\lambda} L \), we see that the above equation is equivalent to \( x' \tanh(x') + L = 0 \) but we can easily verify that \( x' \tanh(x') \geq 0 \) for all \( x' \) so this case cannot happen.

- \( \lambda < 0 \): We have \( v(x) = B \sin(\sqrt{\lambda} x) + C \cos(\sqrt{\lambda} x) \). Using \( u_x(0,t) = 0 \) means that \( B = 0 \). We now use the homogenous mixed condition (again treat \( c(t) \) as a constant).

\[ -C \sqrt{\lambda} \sin(\sqrt{\lambda} L) + C \cos(\sqrt{\lambda} L) = 0. \]

The above equation implies that \( \sqrt{\lambda} \) must satisfy the equation \( 1 = t \tan(tL) \) so let \( \mu_k \) be the \( k \)th positive solution to \( 1 = t \tan(tL) \). Then we must have \( \lambda = -\mu_k^2 \) so we can write

\[ u(t,x) = U + \sum_{k=1}^{\infty} c_k e^{-D \mu_k^2 t} \cos(\mu_k x) \]

for some constants \( c_k \). Now our task is to show that if \( k \neq j \), we actually have \( \langle \cos(\mu_k x), \cos(\mu_j x) \rangle = 0 \). Indeed, if \( k \neq j \),

\[
\int_0^L \cos(\mu_k x) \cos(\mu_j x) \, dx = \frac{1}{2} \int_0^L \cos((\mu_k + \mu_j) x) \, dx + \frac{1}{2} \int_0^L \cos((\mu_k - \mu_j) x) \, dx
\]

\[
= \frac{1}{2} \left( \frac{\sin((\mu_k + \mu_j)L)}{\mu_k + \mu_j} + \frac{\sin((\mu_k - \mu_j)L)}{\mu_k - \mu_j} \right)
\]

\[
= \frac{\mu_k \sin(\mu_k L) \cos(\mu_j L) - \mu_j \sin(\mu_j L) \cos(\mu_k L)}{\mu_k^2 - \mu_j^2}
\]

\[
= \frac{\cos(\mu_j L) \cos(\mu_k L)}{\mu_k^2 - \mu_j^2} \left( \frac{\sin(\mu_k L)}{\cos(\mu_k L)} - \frac{\sin(\mu_j L)}{\cos(\mu_j L)} \right)
\]

\[
= \frac{\cos(\mu_j L) \cos(\mu_k L)}{\mu_k^2 - \mu_j^2} \left( \mu_k \tan(\mu_k L) - \mu_j \tan(\mu_j L) \right).
\]

But note that \( \mu_k \) and \( \mu_j \) are both solutions to the equation \( 1 = t \tan(tL) \). Hence, the expression above is 0 and we indeed have \( \langle \cos(\mu_k x), \cos(\mu_j x) \rangle = 0 \). This means that can compute the \( c_k \) coefficients:

\[ c_k = \frac{1}{\alpha_k} \int_0^L (g(x) - U) \cos(\mu_k x) \, dx \]

where \( \frac{1}{\alpha_k} \) is a normalizing constant that satisfies

\[ \alpha_k = \int_0^L \cos(\mu_k x)^2 \, dx \]

and we are done.
3. Problem 3

We divide \( u(t, x) \) into several parts. Let

\[
u_N(t, x) = \sum_{m=1}^{N} (-1)^{m+1} e^{-m^2 \pi^2 t} \frac{2}{m \pi} \sin(m \pi x).
\]

We know from "Some basic facts" that

\[
\|x - u_N(0, x)\|_{L^2([0,1])} \to 0 \text{ as } N \to \infty.
\]

Fix \( N \) large. We have

\[
\|u_N(t, x) - u_N(0, x)\|_{L^2([0,1])} \to 0 \text{ as } t \to 0.
\]

To justify this, note that \( u_N(t, x) \) is continuous in \([0, 1] \times [0, 1]\), hence uniformly continuous. So as \( t \to 0 \), \( \|u_N(t, x) - u_N(0, x)\|_{C^0([0,1])} \to 0 \), which implies that \( \|u_N(t, x) - u_N(0, x)\|_{L^2([0,1])} \to 0 \). Finally, we want to bound

\[
\|u_N(t, x) - u(t, x)\|_{L^2([0,1])}.
\]

By Parseval's identity,

\[
\|u_N(t, x) - u(t, x)\|^2_{L^2([0,1])} = \frac{1}{2} \sum_{m>N} e^{-2m^2 \pi^2 t} \frac{4}{m^2 \pi^2} \leq \frac{1}{2} \sum_{m>N} \frac{4}{m^2 \pi^2} \to 0
\]

as \( N \to \infty \). Note that this is because \( \sum (1/m^2) \) is finite.
Putting this all together, we fix $\varepsilon > 0$. First choose $N$ so that
\[ \|x - u_N(0, x)\|_{L^2([0,1])} < \frac{\varepsilon}{3}, \quad \|u_N(t, x) - u(t, x)\|_{L^2([0,1])} < \frac{\varepsilon}{3} \]
regardless of $t$. Then for $t > 0$ small,
\[ \|u_N(t, x) - u_N(0, x)\|_{L^2([0,1])} < \frac{\varepsilon}{3}. \]
Therefore, for $t > 0$ small,
\[ \|u(t, x) - x\| \leq \|x - u_N(0, x)\| + \|u_N(t, x) - u_N(0, x)\| + \|u_N(t, x) - u(t, x)\| < \varepsilon, \]
which is what we want.

4. PROBLEM 4

First, we directly compute
\[ \|u(0, \cdot)\|_{L^2([0,\ell])} = \left( \int_0^\ell \frac{x^2(\ell - x)^2}{\ell^4} \, dx \right)^{1/2} = \left( \ell \int_0^1 x^2(1 - x)^2 \, dx \right)^{1/2} = \sqrt{\frac{\ell}{30}}. \]
Now
\[ \frac{d}{dt} \|u(t, \cdot)\|_{L^2([0,\ell])}^2 = \frac{d}{dt} \int_0^\ell u(t, x)^2 \, dx = \int_0^\ell \partial_t(u(t, x)^2) \, dx \]
\[ = \int_0^\ell 2u(t, x)\partial_t u(t, x) \, dx = 2 \int_0^\ell u(t, x)\partial_{xx} u(t, x) \, dx. \]
Differentiation under the integral sign is justified because $\partial_t u \in C(\overline{S})$. Now we integrate by parts to get
\[ \int_0^\ell u(t, x)\partial_{xx} u(t, x) \, dx = u(t, x)\partial_x u(t, x)|_{x=0}^\ell - \int_0^\ell \partial_x u(t, x)^2 \, dx = -\|\partial_x u(t, \cdot)\|_{L^2([0,\ell])}^2, \]
where we use the boundary conditions. We conclude that
\[ (4.1) \quad \frac{d}{dt} \|u(t, \cdot)\|_{L^2([0,\ell])}^2 = -2 \|\partial_x u(t, \cdot)\|_{L^2([0,\ell])}^2. \]
Next, note that for every $0 \leq x \leq \ell$, by Cauchy-Schwarz,
\[ |u(t, x)| = \left| \int_0^x \partial_x u(t, y) \, dy + u(t, 0) \right| \leq \left( \int_0^x 1 \, dy \right)^{1/2} \left( \int_0^x \partial_x u(t, y)^2 \, dy \right)^{1/2} \]
\[ \leq \sqrt{\ell} \left( \int_0^x \partial_x u(t, y)^2 \, dy \right)^{1/2} = \sqrt{\ell} \|\partial_x u(t, \cdot)\|_{L^2([0,\ell])}. \]
Therefore
\[ \|u(t, \cdot)\|_{L^2([0,\ell])}^2 = \int_0^\ell u(t, x)^2 \, dx \leq \ell^2 \|\partial_x u(t, \cdot)\|_{L^2([0,\ell])}^2. \]
Using this on (4.1), we conclude
\[ \frac{d}{dt} \|u(t, \cdot)\|_{L^2([0,\ell])}^2 \leq -\frac{2}{\ell^2} \|u(t, \cdot)\|_{L^2([0,\ell])}^2. \]
Finally, let \( f(t) = \|u(t, \cdot)\|_{L^2([0, \ell])}^2 \). Then \( f'(t) \leq -(2/\ell^2)f(t) \) and \( f(0) = \ell/30 \). This means that
\[
\left(\log f(t)\right)' = \frac{f'(t)}{f(t)} \leq -\frac{2}{\ell^2},
\]
so \( \log f(t) \leq \log(\ell/30) - 2t/\ell^2 \), so \( f(t) \leq (\ell/30)e^{-2t/\ell^2} \). Hence, for all \( t \geq 0 \),
\[
\|u(t, \cdot)\|_{L^2([0, \ell])} = \sqrt{f(t)} \leq \sqrt{\frac{\ell}{30}}e^{-t/\ell^2}.
\]

5. PROBLEM 5

Let \( u(t, x) = V(\zeta)/(Dt)^{1/2} \). Then
\[
\partial_t u = \frac{1}{(Dt)^{1/2}} V'(\zeta) \left( -\frac{\zeta}{2t} \right) + V(\zeta) \left( -\frac{1}{2D^{1/2}\ell^{3/2}} \right) = -\frac{1}{2D^{1/2}\ell^{3/2}} (V(\zeta) + \zeta V'(\zeta))
\]
and
\[
\partial_x^2 u = \frac{1}{(Dt)^{1/2}} V''(\zeta) \frac{1}{Dt} = \frac{V''(\zeta)}{D^{5/2}\ell^{3/2}}.
\]
Therefore
\[
\partial_t u - \partial_x^2 u = \frac{1}{D^{1/2}\ell^{3/2}} \left( -\frac{1}{2}(\zeta V(\zeta))' - V''(\zeta) \right) = 0.
\]

So \( V \) must satisfy the ODE
\[
\frac{d}{d\zeta} \left( V'(\zeta) + \frac{1}{2}\zeta V(\zeta) \right) = 0.
\]

Then \( u(t, x) = u(t, -x) \) implies that \( V(\zeta) = V(-\zeta) \), so \( V'(0) = 0 \) since \( V \) is an even function. Moreover, \( \lim_{x \to \pm\infty} u(t, x) = 0 \) implies that \( \lim_{\zeta \to \pm\infty} V(\zeta) = 0 \).

Now from (5.1), we know that \( V'(\zeta) + \frac{1}{2}\zeta V(\zeta) \) is constant, and because \( V'(0) = 0 \), this constant must be 0. So
\[
V'(\zeta) + \frac{1}{2}\zeta V(\zeta) = 0.
\]

We now find that
\[
(\log V(\zeta))' = \frac{V'(\zeta)}{V(\zeta)} = -\frac{1}{2}\zeta.
\]

By integrating, \( V(\zeta) = V(0)e^{-\zeta^2/4} \). This means that
\[
u(t, x) = \frac{1}{(Dt)^{1/2}} V(0)e^{-x^2/(4Dt)}.\]

Finally, the second demand implies
\[
1 = \int_{\mathbb{R}} u(t, x) \, dx = \frac{V(0)}{(Dt)^{1/2}} \int_{\mathbb{R}} e^{-x^2/(4Dt)} \, dx = \frac{V(0)}{(Dt)^{1/2}} \cdot 2(Dt)^{1/2} \int_{\mathbb{R}} e^{-y^2} \, dy = 2V(0)\sqrt{\pi},
\]
where we use the change of variables \( x = 2(Dt)^{1/2}y \) and the Gaussian integral \( \int_{\mathbb{R}} e^{-y^2} \, dy = \sqrt{\pi} \). Therefore \( V(0) = 1/\sqrt{4\pi} \). We conclude that a solution satisfying the four demands is
\[
u(t, x) = \frac{1}{(4\pi D t)^{1/2}} e^{-x^2/(4Dt)}.\]