I. Consider the global Cauchy problem for the wave equation in $\mathbb{R}^{1+n}$:

\begin{align}
(0.0.1a) & \quad -\partial_t^2 u(t, x) + \Delta u(t, x) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n, \\
(0.0.1b) & \quad u(0, x) = f(x), \\
(0.0.1c) & \quad \partial_t u(0, x) = g(x).
\end{align}

Let the vectorfield $J(t, x)$ on $\mathbb{R}^{1+n}$ be defined as follows:

\begin{equation}
(0.0.2) \quad J = (J^0, J^1, \ldots, J^n) \overset{\text{def}}{=} \left( \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2, -\partial_1 u \partial_t u, -\partial_2 u \partial_t u, \ldots, -\partial_n u \partial_t u \right).
\end{equation}

Above, $x = (x^1, \ldots, x^n)$ denotes coordinates on $\mathbb{R}^n$, $\nabla u \overset{\text{def}}{=} (\partial_1 u, \ldots, \partial_n u)$ is the spatial gradient of $u$, and $|\nabla u|^2 \overset{\text{def}}{=} \sum_{i=1}^n (\partial_i u)^2$ is the square of its Euclidean length.

a) First show that

\begin{equation}
(0.0.3) \quad \partial_t J^0 + \sum_{i=1}^n \partial_i J^i = 0
\end{equation}

whenever $u$ is a $C^2$ solution to (0.0.1a).

b) Then show that if $V = (V^0, V^1, \ldots, V^n) = (1, \omega^1, \omega^2, \ldots, \omega^n) \in \mathbb{R}^{1+n}$ is any vector with $\sum_{i=1}^n (\omega_i)^2 \leq 1$, then

\begin{equation}
(0.0.4) \quad V \cdot J \overset{\text{def}}{=} \sum_{\mu=0}^n J^\mu V^\mu \geq 0.
\end{equation}

**Hint:** To get started, try using the Cauchy-Schwarz inequality for dot products.

II. Assume that $0 \leq t \leq R$, and let $p \in \mathbb{R}^n$ be a fixed point. Let $C_{t,p;R} \overset{\text{def}}{=} \{(\tau, y) \in [0, t) \times \mathbb{R}^n \mid |y-p| \leq R - \tau\} \subset \mathbb{R}^{1+n}$ be a solid, truncated backwards light cone. Note that the boundary of the cone consists of 3 pieces: $\partial C_{t,p;R} = \mathcal{B} \cup \mathcal{M}_{t,p;R} \cup \mathcal{T}$, where $\mathcal{B} \overset{\text{def}}{=} \{0\} \times B_R(p)$ is the flat base of the truncated cone, $\mathcal{T} \overset{\text{def}}{=} \{t\} \times B_{R-t}(p)$ is the flat top of the truncated cone, and $\mathcal{M}_{t,p;R} \overset{\text{def}}{=} \{(\tau, y) \in [0, t) \times \mathbb{R}^n \mid |y-p| = R - \tau\}$ is the mantle (i.e., the side boundary) of the truncated cone.

Define the energy of a function $u$ at time $t$ on the **solid** ball $B_{R-t}(p)$ by

\begin{equation}
(0.0.5) \quad E(t; R; p) \overset{\text{def}}{=} \int_{B_{R-t}(p)} J^0(t, x) \, d^n x \overset{\text{def}}{=} \frac{1}{2} \int_{B_{R-t}(p)} (\partial_t u)^2 + |\nabla u|^2 \, d^n x,
\end{equation}
and recall that the divergence theorem in $\mathbb{R}^{1+n}$ implies that

\begin{equation}
(0.0.6) \quad \int_{\mathcal{C}_{t,p;R}} \left( \partial_t J^0 + \sum_{i=1}^{n} \partial_i J^i \right) d^n x dt = \int_{\mathcal{M}_{t,p;R}} \mathbf{N}(\sigma) \cdot \mathbf{J} d\sigma - \int_{B_{R}(p)} J^0 d^n x + \int_{B_{R-1}(p)} J^0 d^n x.
\end{equation}

In (0.0.6), $\mathbf{N}(\sigma)$ is the unit outward normal to $\mathcal{M}_{t,p;R}$.

**Remark 0.0.1.** In the near future, we will discuss the geometry of Minkowski spacetime, which is intimately connected to the linear wave equation. Our study will lead to a geometrically motivated construction of the vectorfield $\mathbf{J}$ and the identity (0.0.6). Alternatively, the identity (0.0.6) could also be derived by multiplying both sides of equation (0.0.1a) by $-\partial_t u$, then integrating by parts and using the divergence theorem.

**a)** Show that the unit outward normal $\mathbf{N}(\sigma)$ to $\mathcal{M}_{t,p;R}$ is of the form

\begin{equation}
(0.0.7) \quad \mathbf{N}(\sigma) = \frac{1}{\sqrt{2}} (1, \omega^1(\sigma), \omega^2(\sigma), \cdots, \omega^n(\sigma)),
\end{equation}

where $\sum_{i=1}^{n} (\omega^i)^2 = 1$. Note that by translational invariance, you may assume that $p = 0$.

**b)** With the help of Problem 1 and (0.0.6)-(0.0.7), show that if $u$ is a $C^2$ solution to (0.0.1a), then

\begin{equation}
(0.0.8) \quad E(t; R; p) \leq E(0; R; p)
\end{equation}

holds for all $t$ with $0 \leq t \leq R$.

**c)** Then show that if the initial data functions $f(x)$ and $g(x)$ from (0.0.1b)-(0.0.1c) are both smooth and vanish outside of the ball $B_{R_0}(p) \subset \mathbb{R}^n$, then at each time $t \geq 0$, the solution $u(t, x)$ to (0.0.1a) vanishes outside of the ball $B_{R_0+t}(p)$.

**d)** Finally, under the same assumptions on $f$ and $g$, let $R \to \infty$ in (0.0.8) (and also use additional arguments) to show that the solution $u$ to (0.0.1a) satisfies

\begin{equation}
(0.0.9) \quad \left\| \nabla_{t,x} u(t, \cdot) \right\|_{L^2(\mathbb{R}^n)} = \left\| \nabla_{t,x} u(0, \cdot) \right\|_{L^2(\mathbb{R}^n)} \overset{\text{def}}{=} \left( \int_{\mathbb{R}^n} |g(x)|^2 + |\nabla f(x)|^2 d^n x \right)^{1/2},
\end{equation}

where $\nabla_{t,x} u = (\partial_t u, \partial_1, \cdots, \partial_n u)$ is the spacetime gradient of $u$, $\left\| \nabla_{t,x} u \right\| \overset{\text{def}}{=} \sqrt{(\partial_t u)^2 + (\partial_1 u)^2 + \cdots + (\partial_n u)^2}$, and the $L^2$ norms in (0.0.9) are taken over the spatial variables only.

**III.** Let $R > 0$, and let $f(x)$, $g(x)$ be smooth functions on $\mathbb{R}$ that vanish outside of $B_R(0) \overset{\text{def}}{=} [-R, R]$. Let $u(t, x)$ be the corresponding unique solution to the following global Cauchy problem on $\mathbb{R}^{1+1}$:

\begin{align}
(0.0.10a) \quad &-\partial_t^2 u(t, x) + \partial_x^2 u(t, x) = 0, \\
(0.0.10b) \quad &u(0, x) = f(x), \\
(0.0.10c) \quad &\partial_t u(0, x) = g(x).
\end{align}
We define the following quantities:

(0.0.11a) \[ P(t) \overset{\text{def}}{=} \int_{\mathbb{R}} (\partial_x u(t, x))^2 \, dx, \quad \text{the potential energy} \]

(0.0.11b) \[ K(t) \overset{\text{def}}{=} \int_{\mathbb{R}} (\partial_t u(t, x))^2 \, dx, \quad \text{the kinetic energy} \]

(0.0.11c) \[ E(t) \overset{\text{def}}{=} P(t) + K(t), \quad \text{the total energy}. \]

In Problem II, you used energy methods to prove that \( E(t) \) is conserved: \( E(t) = E(0) \) for all \( t \geq 0 \). Now show that if \( t \) is large enough, then \( P(t) = K(t) = \frac{1}{2} E(t) \). This is called the equipartitioning of the energy.

**Hint:** Try expressing \( P(t) \) and \( K(t) \) in terms of the null derivatives \( \partial_q u(t, x) \) and \( \partial_s u(t, x) \) that we used in the proof of d’Alembert’s formula. If you set up the calculations properly, then the desired equipartitioning result should boil down to proving that \( \int_{\mathbb{R}} (\partial_q u(t, x))(\partial_s u(t, x)) \, dx = 0 \) for all large \( t \). In order to prove this latter result, take a close look at the the expressions for \( \partial_q u(t, x) \) and \( \partial_s u(t, x) \) that we derived in terms of \( f, g \) during that proof, and make use of the assumptions on \( f, g \).