1. Representation Formula for Solutions to Poisson’s Equation

We now derive our main representation formula for solutions to Poisson’s equation on a domain \( \Omega \).

**Theorem 1.1 (Representation formula for solutions to the boundary value Poisson equation).** Let \( \Omega \) be a domain with a smooth boundary, and assume that \( f \in C^2(\overline{\Omega}) \) and \( g \in C(\partial \Omega) \). Then the unique solution \( u \in C^2(\Omega) \cap C(\Omega) \) to

\[
\Delta u(x) = f(x), \quad x \in \Omega \subset \mathbb{R}^n, \\
u(x) = g(x), \quad x \in \partial \Omega.
\]

(1.0.1)

can be represented as

\[
u(x) = \int_{\Omega} f(y) G(x,y) \, d^n y + \int_{\partial \Omega} g(\sigma) \nabla \tilde{N}(\sigma) G(x,\sigma) \, d\sigma,\]

(1.0.2)

where \( G(x,y) \) is the Green function for \( \Omega \).

**Proof.** Applying the Representation formula for \( u \) Proposition, we have that

\[
u(x) = \int_{\Omega} \Phi(x-y) f(y) \, d^n y - \int_{\partial \Omega} \Phi(x-\sigma) \nabla \tilde{N}(\sigma) u(\sigma) \, d\sigma + \int_{\partial \Omega} g(\sigma) \nabla \tilde{N}(\sigma) \Phi(x-\sigma) \, d\sigma.
\]

(1.0.3)

Recall also that

\[
G(x,y) = \Phi(x-y) - \phi(x,y),
\]

(1.0.4)

where

\[
\Delta_y \phi(x,y) = 0, \quad x \in \Omega,
\]

(1.0.5)

and

\[
G(x,\sigma) = 0 \text{ when } x \in \Omega \text{ and } \sigma \in \partial \Omega.
\]

(1.0.6)
Remark 2.0.1.

The expression (1.0.3) is not very useful since don’t know the value of \( \nabla_N u(\sigma) \) along \( \partial \Omega \). To fix this, we will use Green’s identity. Applying Green’s identity to the functions \( u(y) \) and \( \phi(x, y) \), and recalling that \( \Delta_y \phi(x, y) = 0 \) for each fixed \( x \in \Omega \), we have that

\[
0 = \int_{\Omega} \phi(x, y) f(y) \, d^n y - \int_{\partial \Omega} \left( \phi(x, \sigma) \nabla_N u(\sigma) + g(\sigma) \nabla_N \phi(x, \sigma) \right) \, d\sigma.
\]

Subtracting (1.0.7) from (1.0.3), and using (1.0.4), we deduce the formula (1.0.2).

2. Poisson’s Formula

Let’s compute the Green function \( G(x, y) \) and Poisson kernel \( P(x, \sigma) \) \( \overset{\text{def}}{=} \nabla_N G(x, \sigma) \) from (1.0.2) in the case that \( \Omega \overset{\text{def}}{=} B_R(0) \subset \mathbb{R}^3 \) is a ball of radius \( R \) centered at the origin. We’ll use a technique called the method of images that works for special domains.

Warning 2.0.1. Brace yourself for a bunch of tedious computations that at the end of the day will lead to a very nice expression.

The basic idea is to hope that \( \phi(x, y) \) from the decomposition \( G(x, y) = \Phi(x - y) - \phi(x, y) \), where \( \phi(x, y) \) is viewed as a function of \( x \) that depends on the parameter \( y \), is equal to the Newtonian potential generated by some “imaginary charge” \( q \) placed at a point \( x^* \in B_R^c(0) \). To ensure that \( G(x, \sigma) = 0 \) when \( \sigma \in \partial B_R(0) \), \( q \) and \( x^* \) have to be chosen so that along the boundary \( \{ y \in \mathbb{R}^3 \mid |y| = R \} \), \( \phi(x, y) = \frac{1}{4\pi|x-y|} \). In a nutshell, we guess that

\[
G(x, y) = -\frac{1}{4\pi|x-y|} + \frac{q}{4\pi|x^*-y|} - \phi(x,y)\tag{2.0.1}
\]

and we try to solve for \( q \) and \( x^* \) so that \( G(x, y) \) vanishes when \( |y| = R \).

Remark 2.0.1. Note that \( \Delta_y \frac{q}{4\pi|x^*-y|} = 0 \) for \( x, y \in B_R(0) \), which is one of the conditions necessary for constructing \( G(x, y) \).

By the definition of \( G(x, y) \), we must have \( G(x, y) = 0 \) when \( |y| = R \), which implies that

\[
\frac{1}{4\pi|x-y|} = \frac{q}{4\pi|x^*-y|} \tag{2.0.2}
\]

Simple algebra then leads to the identity

\[
|x^*-y|^2 = q^2|x-y|^2 \tag{2.0.3}
\]

When \( |y| = R \), we use (2.0.3) to compute that

\[
x^2 + 2x^* \cdot y + R^2 = |x^*-y|^2 = q^2|x-y|^2 = q^2(|x|^2 - 2x \cdot y + R^2), \tag{2.0.4}
\]
where \( \cdot \) denotes the Euclidean dot product. Then performing simple algebra, we deduce from (2.0.4) that

\[
|x^*|^2 + R^2 - q^2(R^2 + |x|^2) = 2y \cdot (x^* - q^2 x). \tag{2.0.5}
\]

Now since the left-hand side of (2.0.5) does not depend on \( y \), it must be the case that the right-hand side is always 0. This implies that \( x^* = q^2 x \), and also leads to the equation

\[
q^4|x|^2 - q^2(R^2 + |x|^2) + R^2 = 0. \tag{2.0.6}
\]

Solving (2.0.6) for \( q \), we finally have that

\[
q = \frac{R}{|x|}, \tag{2.0.7}
\]

\[
x^* = \frac{R^2}{|x|^2} x. \tag{2.0.8}
\]

Therefore,

\[
\phi(x, y) = -\frac{1}{4\pi} \frac{R}{|x| |x^* - y|}, \tag{2.0.9}
\]

\[
\phi(0, y) = -\frac{1}{4\pi R}, \tag{2.0.10}
\]

where we took a limit as \( x \to 0 \) in (2.0.9) to derive (2.0.10).

Next, using (2.0.1), we have

\[
G(x, y) = -\frac{1}{4\pi|x-y|} + \frac{1}{4\pi} \frac{R}{|x| |x^* - y|}, \quad x \neq 0, \tag{2.0.11}
\]

\[
G(0, y) = -\frac{1}{4\pi|y|} + \frac{1}{4\pi R}. \tag{2.0.12}
\]

For future use, we also compute that

\[
\nabla_y G(x, y) = -\frac{x-y}{4\pi |x-y|^3} + \frac{1}{4\pi} \frac{R}{|x| |x^*-y|^3}. \tag{2.0.13}
\]

Now when \( \sigma \in \partial B_R(0) \), (2.0.3) and (2.0.7) imply that

\[
|x^* - \sigma| = \frac{R}{|x|} |x - \sigma|. \tag{2.0.14}
\]

Therefore, using (2.0.13) and (2.0.14), we compute that

\[
\nabla_{\hat{N}(\sigma)} G(x, \sigma) = -\frac{x - \sigma}{4\pi |x - \sigma|^3} + \frac{1}{4\pi} \frac{|x|^2}{R^2} \frac{x^* - \sigma}{|x - \sigma|^3} = -\frac{x - \sigma}{4\pi |x - \sigma|^3} + \frac{1}{4\pi} \frac{|x|^2}{R^2} \frac{R^2}{|x - \sigma|^3} = \frac{\sigma}{4\pi |x - \sigma|^3} \left( 1 - \frac{|x|^2}{R^2} \right). \tag{2.0.15}
\]
Using (2.0.15) and the fact that \( \hat{N}(\sigma) = \frac{1}{R} \sigma \), we deduce

\[
\nabla_{\hat{N}(\sigma)} G(x,\sigma) \overset{\text{def}}{=} \nabla_\sigma G(x,\sigma) \cdot \hat{N}(\sigma) = \frac{R^2 - |x|^2}{4\pi R} \frac{1}{|x-\sigma|^3}. \tag{2.0.16}
\]

**Remark 2.0.2.** If the ball were centered at the point \( p \in \mathbb{R}^3 \) instead of the origin, then the formula (2.0.16) would be replaced with

\[
\nabla_{\hat{N}(\sigma)} G(x,\sigma) \overset{\text{def}}{=} \nabla_\sigma G(x,\sigma) \cdot \hat{N}(\sigma) = -\frac{R^2 - |x-p|^2}{4\pi R} \frac{1}{|x-\sigma|^3}. \tag{2.0.17}
\]

In the following lemma, we summarize the above results.

**Lemma 2.0.1.** The Green function for a ball \( B_R(p) \subset \mathbb{R}^3 \) is

\[
G(x,y) = -\frac{1}{4\pi|x-y|} + \frac{1}{4\pi|x-p|} \frac{R}{|x-p|^2(x-p) \cdot (y-p)} \quad \text{for} \quad x \neq p, \tag{2.0.18a}
\]

\[
G(p,y) = -\frac{1}{4\pi|y-p|} + \frac{1}{4\pi R}. \tag{2.0.18b}
\]

Furthermore, if \( x \in B_R(p) \) and \( \sigma \in \partial B_R(p) \), then

\[
\nabla_{\hat{N}(\sigma)} G(x,\sigma) \overset{\text{def}}{=} \nabla_\sigma G(x,\sigma) \cdot \hat{N}(\sigma) = \frac{R^2 - |x-p|^2}{4\pi R} \frac{1}{|x-\sigma|^3}. \tag{2.0.18c}
\]

We can now easily derive a representation formula for solutions to the Laplace equation on a ball.

**Theorem 2.1 (Poisson’s formula).** Let \( B_R(p) \subset \mathbb{R}^3 \) be a ball of radius \( R \) centered at \( p = (p^1, p^2, p^3) \), and let \( x = (x^1, x^2, x^3) \) denote a point in \( \mathbb{R}^3 \). Let \( g \in C(\partial B_R(p)) \). Then the unique solution \( u \in C^2(B_R(p)) \cap C(\overline{B_R(p)}) \) of the PDE

\[
\Delta u(x) = 0, \quad u(x) = g(x), \quad x \in \partial B_R(p), \tag{2.0.19}
\]

can be represented using the Poisson formula:

\[
\[ u(x) = \frac{R^2 - |x-p|^2}{4\pi R} \int_{\partial B_R(p)} \frac{g(\sigma)}{|x-\sigma|^3} d\sigma. \tag{2.0.20} \]
\]

**Remark 2.0.3.** In \( n \) dimensions, the formula (2.0.20) gets replaced with

\[
u(x) = \frac{R^2 - |x-p|^2}{\omega_n R} \int_{\partial B_R(p)} \frac{g(\sigma)}{|x-\sigma|^n} d\sigma, \tag{2.0.21}\]

where as usual, \( \omega_n \) is the surface area of the unit ball in \( \mathbb{R}^n \).

**Proof.** The identity (2.0.20) follows immediately from Theorem 1.1 and Lemma 2.0.1. \( \square \)
3. Harnack’s inequality

We will now use some of our tools to prove a famous inequality for harmonic functions. The theorem provides some estimates that place limitations on how slow/fast harmonic functions are allowed to grow.

**Theorem 3.1 (Harnack’s inequality).** Let $B_R(0) \subset \mathbb{R}^n$ be the ball of radius $R$ centered at the origin, and let $u \in C^2(B_R(0)) \cap C(\overline{B_R(0)})$ be the unique solution to (2.0.19). Assume that $u$ is non-negative on $B_R(0)$. Then for any $x \in B_R(0)$, we have that

$$u(x) \leq R^{n-2}(R - |x|) u(0) \leq u(x) \leq R^{n-2}(R + |x|) u(0).$$

**Proof.** We’ll do the proof for $n = 3$. The basic idea is to combine the Poisson representation formula with simple inequalities and the mean value property. By Theorem 2.1, we have that

$$u(x) = \frac{R^2 - |x|^2}{4\pi R} \int_{\partial B_R(0)} \frac{g(\sigma)}{|x - \sigma|^3} d\sigma.$$  

By the triangle inequality, for $\sigma \in \partial B_R(0)$ (i.e. $|\sigma| = R$), we have that $R - |x| \leq |x - \sigma| \leq |x| + R$. Applying the first inequality to (3.0.2), and using the non-negativity of $g$, we deduce that

$$u(x) \leq \frac{R + |x|}{(R - |x|)^2} \frac{1}{4\pi R} \int_{\partial B_R(0)} g(\sigma) d\sigma.$$  

Now recall that by the mean value property, we have that

$$u(0) = \frac{1}{4\pi R^2} \int_{\partial B_R(0)} g(\sigma) d\sigma.$$  

Thus, combining (3.0.3) and (3.0.4), we have that

$$u(x) \leq \frac{R(R + |x|)}{(R - |x|)^2} u(0),$$

which implies one of the inequalities in (3.0.1). The other one can be proved similarly using the remaining triangle inequality.

□

We now prove a famous consequence of Harnack’s inequality. The statement is also often proved in introductory courses in complex analysis, and it plays a central role in some proofs of the fundamental theorem of algebra.

**Corollary 3.0.1 (Liouville’s theorem).** Suppose that $u \in C^2(\mathbb{R}^n)$ is harmonic on $\mathbb{R}^n$. Assume that there exists a constant $M$ such that $u(x) \geq M$ for all $x \in \mathbb{R}^n$, or such that $u(x) \leq M$ for all $x \in \mathbb{R}^n$. Then $u$ is a constant-valued function.
Proof. We first consider the case that \( u(x) \geq M \). Let \( v(x) \defeq u(x) + |M| \). Observe that \( v \geq 0 \) is harmonic and verifies the hypotheses of Theorem 3.1. Thus, by (3.0.1), if \( x \in \mathbb{R}^n \) and \( R \) is sufficiently large, we have that
\[
(3.0.6) \quad \frac{R^{n-2}(R - |x|)}{(R + |x|)^{n-1}} v(0) \leq v(x) \leq \frac{R^{n-2}(R + |x|)}{(R - |x|)^{n-1}} v(0).
\]

Allowing \( R \to \infty \) in (3.0.6), we conclude that \( v(x) = v(0) \). Thus, \( v \) is a constant-valued function (and therefore \( u \) is too).

To handle the case \( u(x) \leq M \), we simply consider the function \( w(x) \defeq -u(x) + |M| \) in place of \( v(x) \), and we argue as above.

\( \square \)