I. Problem 2.3 on pg. 109.

II. Problem 2.4 on pg. 109.

III. Consider the solution $u(t, x) = \sum_{m=1}^{\infty} (-1)^{m+1} e^{-m^2 \pi^2 t} \frac{2}{m \pi} \sin(m \pi x)$ to the initial-boundary value heat equation problem

$$\begin{cases}
\partial_t u - \partial_x^2 u = 0, & (t, x) \in (0, \infty) \times (0, 1), \\
u(0, x) = x, & x \in [0, 1], \\
u(t, 0) = u(t, 1) = 0, & t \in (0, \infty),
\end{cases}$$

as discussed in class. Show that

$$\lim_{t \downarrow 0} \|u(t, x) - x\|_{L^2([0, 1])} = 0,$$

where the $L^2$ norm is taken over the $x$ variable only. Feel free to make use of the “Some basic facts from Fourier analysis” theorem discussed in class.

Remark 0.0.1. This problem shows that even though there is a pointwise discontinuity at $(0, 1)$, the solution is nonetheless “continuous in $t$ at $t = 0$” with respect to the $L^2([0, 1])$ spatial norm.

IV. Let $\ell > 0$ be a positive real number. Let $S = (0, \infty) \times (0, \ell)$, and let $u(t, x) \in C^{1,2}(\overline{S})$ be the solution of the initial-boundary value problem

$$\begin{cases}
\partial_t u - \partial_x^2 u = 0, & (t, x) \in S, \\
u(0, x) = \ell^2 x(\ell - x), & x \in [0, \ell], \\
u(t, 0) = 0, & u(t, \ell) = 0, & t \in (0, \infty).
\end{cases}$$

In this problem, you will use the energy method to show that the spatial $L^2$ norm of $u$ decays exponentially without actually having to solve the PDE.

First show that $\|u(0, \cdot)\|_{L^2([0,\ell])} = \sqrt{\frac{\ell}{30}}$. Here, the notation $\|u(0, \cdot)\|_{L^2([0,\ell])}$ is meant to emphasize that the $L^2$ norm is taken over the spatial variable $x$ only.

Next, show that $\frac{d}{dt}(\|u(t, \cdot)\|_{L^2([0,\ell])}^2) = -2\|\partial_x u(t, \cdot)\|_{L^2([0,\ell])}^2$.

Then show that $\|u(t, \cdot)\|_{C^0([0,\ell])} \leq \sqrt{7}\|\partial_x u(t, \cdot)\|_{L^2([0,\ell])}$ (Hint: Use the Fundamental theorem of calculus in $x$ and the Cauchy-Schwarz inequality with one of the functions equal to 1).
Then conclude that \( \|u(t, \cdot)\|_{L^2}^2 \leq \ell^2 \|\partial_x u(t, \cdot)\|_{L^2([0,\ell])}^2 \). Using a previous part of this problem, we conclude that \( \frac{d}{dt}(\|u(t, \cdot)\|_{L^2([0,1])}^2) \leq -2\frac{1}{\ell^2} \|u(t, \cdot)\|_{L^2([0,\ell])}^2 \).

Finally, integrate this differential inequality in time and use the initial conditions at \( t = 0 \) to conclude that \( \|u(t, \cdot)\|_{L^2([0,1])} \leq \sqrt{\frac{\ell}{30}} e^{-\frac{t}{\ell^2}} \) for all \( t \geq 0 \).

V. In this problem, you will derive a very important solution to the heat equation on \( \mathbb{R}^{1+1} \):

\[
\partial_t u - D\partial_x^2 u = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}.
\]

The special solution \( u(t, x) \) will be known as the fundamental solution, and it plays a very important role in the theory of the heat equation on \( (0, \infty) \times \mathbb{R} \). We demand that our fundamental solution \( u(t, x) \) should have the following properties:

- \( u(t, x) \geq 0 \)
- \( \int_{\mathbb{R}} u(t, x) \, dx = 1 \) for all \( t > 0 \)
- \( \lim_{x \to \pm \infty} u(t, x) = 0 \) for all \( t > 0 \)
- \( u(t, x) = u(t, -x) \) for all \( t > 0 \)

To see that such a solution exists, first make the assumption that \( u(t, x) = \frac{1}{\sqrt{D}t} V(\xi) \), where \( \xi \overset{\text{def}}{=} \frac{x}{\sqrt{Dt}} \) and \( V(\xi) \) is a function that is (hopefully) defined for all \( \xi \in \mathbb{R} \); we will motivate this assumption in class. Show that if \( u \) verifies (0.0.3), then \( V \) must satisfy the ODE

\[
\frac{d}{d\xi}(V'(\xi) + \frac{1}{2}\xi V(\xi)) = 0.
\]

Then, using the above demands, argue that \( V(\xi) = V(-\xi) \), \( V'(0) = 0 \), and \( \lim_{\xi \to \pm} V(\xi) = 0 \). Also using (0.0.4), argue that

\[
V'(\xi) + \frac{1}{2}\xi V(\xi) = 0.
\]

Integrate (0.0.5) to conclude that \( V(\xi) = V(0)e^{-\frac{\xi^2}{4Dt}} \), which implies that

\[
u(t, x) = \frac{1}{\sqrt{Dt}} V(0)e^{-\frac{x^2}{4Dt}}.
\]

Finally, use the second demand from above to conclude that \( V(0) = \frac{1}{\sqrt{4\pi}} \).