1. The Fundamental Solution for $\Delta$ in $\mathbb{R}^n$

Here is a situation that often arises in physics. We are given a function $f(x)$ on $\mathbb{R}^n$ representing the spatial density of some kind of quantity, and we want to solve the following Poisson equation:

\begin{equation}
\Delta u(x) = f(x), \quad x = (x^1, \ldots, x^n) \in \mathbb{R}^n.
\end{equation}

Furthermore, we often want to impose the following decay condition as $|x| \to \infty$:

\begin{equation}
|u(x)| \to 0.
\end{equation}

For technical reasons, we will need a different condition in the case $n = 2$. A good physical example is the theory of electrostatics, in which $u(x)$ is the electric potential and $f(x)$ is the charge density. $f(x)$ could be e.g. a compactly supported function modeling the charge density of a charged star, and we might want to know how the potential behaves far away from the star (i.e. as $|x| \to \infty$). Roughly speaking, the decay conditions (1.0.2) are physically motivated by the fact that the star should not have a large effect on far-away locations.

As we will soon see, the PDE (1.0.1) has a unique solution verifying (1.0.2) as long as $f(x)$ is sufficiently differentiable and decays sufficiently rapidly as $|x| \to \infty$. Much like in the case of the heat equation, we will be able to construct the solution using an object called the **fundamental solution**.

**Definition 1.0.1.** The fundamental solution $\Phi$ corresponding to the operator $\Delta$ is

\begin{equation}
\Phi(x) \overset{\text{def}}{=} \begin{cases}
\frac{1}{2\pi} \ln |x| & n = 2, \\
\frac{1}{\omega_n |x|^{n-2}} & n \geq 3,
\end{cases}
\end{equation}

where as usual $|x| \overset{\text{def}}{=} \sqrt{\sum_{i=1}^n (x^i)^2}$ and $\omega_n$ is the surface area of a unit ball in $\mathbb{R}^n$ (e.g. $\omega_3 = 4\pi$).

**Remark 1.0.1.** Some people prefer to define their $\Phi$ to be the negative of our $\Phi$.

Essentially, our goal in this section is to show that $\Delta \Phi(x) = \delta(x)$, where $\delta$ is the delta distribution. Let’s assume that this holds for now. We then claim that the solution to (1.0.1) is $u(x) = f \ast \Phi(x) = \int_{\mathbb{R}^n} f(y) \Phi(x-y) \, d^n y$. This can be heuristically justified by the following heuristic computations:

$\Delta_x (f \ast \Phi) = f \ast \Delta_x \Phi = f \ast \delta = f(x)$.

Let’s now make rigorous sense of this. We first show that away from the origin, the fundamental solution verifies Laplace’s equation.

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1Recall that the force $F$ associated to $u$ is $F = -\nabla u$.}
Lemma 1.0.1. If \( x \neq 0 \), then \( \Delta \Phi(x) = 0 \).

Proof. Let’s do the proof in the case \( n = 3 \). Note that \( \Phi(x) = \Phi(r) \) \((r \overset{\text{def}}{=} |x|)\) is spherically symmetric. Thus, using the fact that \( \Delta = \partial_r^2 + \frac{2}{r} \partial_r \) in spherical symmetry, we have that \( \Delta \Phi = \partial_r^2 \Phi + \frac{2}{r} \partial_r \Phi = \frac{-2}{\omega_{3n}} + \frac{2}{\omega_{3n}} = 0 \).

We are now ready to state and prove a rigorous version of the aforementioned heuristic results.

Theorem 1.1 (Solution to Poisson’s equation in \( \mathbb{R}^n \)). Let \( f(x) \in C_0^\infty(\mathbb{R}^n) \) \((\text{i.e., } f(x) \text{ is a smooth, compactly supported function on } \mathbb{R}^n)\). Then for \( n \geq 3 \), Poisson’s equation \( \Delta u(x) = f(x) \) has a unique smooth solution \( u(x) \) that tends to 0 as \( |x| \to \infty \). For \( n = 2 \), the solution is unique under the assumptions \( u(x) \to 0 \) as \( |x| \to \infty \) and \( |\nabla u(x)| \to 0 \) as \( |x| \to \infty \). Furthermore, these unique solutions can be represented as

\[
(1.0.4) \quad u(x) = (\Phi * f)(x) = \begin{cases}
\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |y| f(x-y) \, dy, & n = 2, \\
-\frac{1}{\omega_n} \int_{\mathbb{R}^n} |y|^{-n-2} f(x-y) \, dn, & n \geq 3.
\end{cases}
\]

Furthermore, there exist constants \( C_n > 0 \) such that the following decay estimate holds for the solution as \( |x| \to \infty \):

\[
(1.0.5) \quad |u(x)| \leq \begin{cases}
C_2 \ln |x| & n = 2, \\
\frac{C_n}{|x|^{n-2}} & n \geq 3.
\end{cases}
\]

Remark 1.0.2. As we alluded to above, Theorem 1.1 shows that \( \Delta \Phi(x) = \delta(x) \), where \( \delta \) is the “delta distribution.” For on the one hand, as we have previously discussed, we have that \( f = \delta * f \). On the other hand, our proof of Theorem 1.1 below will show that \( f = \Delta u = \Delta (\Phi * f) = (\Delta \Phi) * f \). Thus, for any \( f \), we have \( \delta * f = (\Delta \Phi) * f \), and so \( \Delta \Phi = \delta \).

Proof. We consider only the case \( n = 3 \). Let’s first show existence by checking that the function \( u \) defined in (1.0.4) solves the equation and has the desired properties. We first differentiate under the integral (we use one of our prior propositions to justify this) and use the fact that \( \Delta \Phi \) defined in (1.0.4) solves the equation and has the desired properties. We first differentiate under the assumptions \( u(x) \to 0 \) as \( |x| \to \infty \) and \( |\nabla u(x)| \to 0 \) as \( |x| \to \infty \). Furthermore, these unique solutions can be represented as

\[
\Delta u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|y|} \Delta_y f(x-y) \, dy = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|y|} \Delta_y f(x-y) \, dy.
\]

To show that the right-hand side of (1.0.6) is equal to \( f(x) \), we will split the integral into two pieces: a small ball centered at the origin, and it’s complement. Thus, let \( B_\epsilon(0) \) denote the ball of radius \( \epsilon \) centered at 0. We then split

\[
\Delta u(x) = -\frac{1}{4\pi} \int_{B_\epsilon(0)} \frac{1}{|y|} \Delta_y f(x-y) \, dy - \frac{1}{4\pi} \int_{B_\epsilon^c(0)} \frac{1}{|y|} \Delta_y f(x-y) \, dy \overset{\text{def}}{=} I + II.
\]

We first show that \( I \) goes to 0 as \( \epsilon \to 0^+ \). To this end, let

\[
M \overset{\text{def}}{=} \sup_{y \in \mathbb{R}^3} |f(y)| + |\nabla f(y)| + |\Delta y f(y)|.
\]
Then using spherical coordinates \((r, \omega)\) for the \(y\) variable, and recalling that \(d^3y = r^2 d\omega\) (where \(\omega \in \partial B_1(0) \subset \mathbb{R}^3\) is a point on the unit sphere and \(d\omega = \sin \theta d\theta d\phi\)) in spherical coordinates, we have that

\[
|I| \leq \int_{B_r(0)} \frac{1}{|y|} \Delta_y f(x-y) \, d^3y \leq M \int_{r=0}^{\epsilon} \int_{\partial B_1(0)} r \, d\omega \, dr = 2\epsilon^2 \pi M.
\]

Clearly, the right-hand side of (1.0.9) goes to 0 as \(\epsilon \to 0^+\).

We would now like to understand the second term on the right-hand side of (1.0.7). We claim that

\[
|f(x) - II| \to 0
\]

as \(\epsilon \to 0^+\). After we show this, we can combine (1.0.7), (1.0.9), and (1.0.10) and let \(\epsilon \to 0^+\) to deduce that \(\Delta_x u(x) = f(x)\) as desired.

To show (1.0.10), we will use integration by parts via Green’s identity and simple estimates to control the boundary terms. Recall that Green’s identity for two functions \(v, w\) is

\[
\int_\Omega v(x) \Delta w(x) - w(x) \Delta v(x) \, d^3x = \int_{\partial\Omega} v \nabla N w(\sigma) - w \nabla N v(\sigma) \, d\sigma.
\]

Using (1.0.11) and Lemma 1.0.1, we compute that

\[
\int_{B_\epsilon(0)} -\frac{1}{|y|} \Delta_y f(x-y) + f(x-y) \frac{1}{|y|} \, d^3y = \int_{\partial B_\epsilon(0)} \frac{1}{|\sigma|} \nabla N(\sigma) f(x-\sigma) - f(x-\sigma) \nabla N(\sigma) \frac{1}{|\sigma|} \, d\sigma.
\]

Above, \(\nabla N(\sigma)\) is the outward unit radial derivative on the sphere \(\partial B_\epsilon(0)\). This corresponds to the “opposite” choice of normal that appears in the standard formulation of Green’s identity for \(B_\epsilon(0)\), but we have compensated for this by carefully inserting minus signs on the right-hand side of (1.0.12). Noting also that \(\nabla N(\sigma) \frac{1}{|\sigma|} = \frac{1}{|\sigma|^2}\), that \(|\sigma| = \epsilon\) on \(\partial B_\epsilon(0)\), and that \(d\sigma = \epsilon^2 d\omega\) on \(\partial B_\epsilon(0)\), we see that

\[
-\int_{B_\epsilon(0)} \frac{1}{|y|} \Delta_y f(x-y) \, d^3y = -\int_{\partial B_\epsilon(0)} \epsilon \omega \cdot (\nabla f)(x-\epsilon \omega) \, d\omega + \int_{\partial B_\epsilon(0)} f(x-\epsilon \omega) \, d\omega.
\]

From (1.0.8), it follows that the first integral on the right-hand side of (1.0.13) is bounded in absolute value by \(4\pi M\epsilon\), and thus goes to 0 as \(\epsilon \to 0^+\). Furthermore, since \(f\) is continuous and since \(\int_{\partial B_\epsilon(0)} 1 \, d\omega = 4\pi\), it follows that the second integral converges to \(4\pi f(x)\) as \(\epsilon \to 0^+\). We have thus proved (1.0.10) for \(n = 3\).

To estimate \(|u(x)|\) as \(|x| \to \infty\), we assume that \(f(x)\) vanishes outside of the ball \(B_R(0)\). It suffices to estimate right-hand side of (1.0.4) when \(|x| > 2R\). We first note the inequality \(\frac{1}{|x-y|} \leq \frac{2}{|x|}\), which holds for \(|y| \leq R\) and \(|x| > 2R\). Using this inequality and (1.0.8), we can estimate right-hand side of (1.0.4) by

\[
|u(x)| = \frac{1}{4\pi} \int_{B_R(0)} \frac{1}{|x-y|} f(y) \, d^3y \leq \frac{M}{2\pi|x|} \int_{B_R(0)} 1 \, d^3y = \frac{2R^3M}{3|x|},
\]

where \(M\) is the maximum of \(f(y)\) on \(B_R(0)\).
and we have shown (1.0.5) in the case $n = 3$.

To prove uniqueness, we will rely on Liouville’s theorem, which we will prove later. If $u, v$ are two solutions with the assumed decay conditions at $\infty$, then using the usual strategy, we note that $w \overset{\text{def}}{=} u - v$ is a solution to Laplace’s equation

\[
\Delta w = 0 \tag{1.0.15}
\]

that verifies $|w(x)| \to 0$ as $|x| \to \infty$. In particular, $w$ is a bounded harmonic function on $\mathbb{R}^3$. We will later prove Liouville’s theorem, which shows that $w(x)$ must be a constant function. Furthermore, the constant must be 0 since $|w(x)| \to 0$ as $|x| \to \infty$. 

$\square$