18.152 Problem Set 1

September 15, 2011

Problem I. Let \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) denote the vector field \( F = u \nabla v - v \nabla u \). Then by the divergence theorem,

\[
\int_{\Omega} \nabla \cdot F \, dx = \int_{\partial \Omega} F \cdot \hat{N} \, d\sigma,
\]

where \( d\sigma \) is the surface measure on \( \partial \Omega \) and \( \hat{N} \) is the unit outward normal vector on \( \partial \Omega \). So

\[
\int_{\partial \Omega} \left( (u(\sigma)) \nabla u(\sigma) v(\sigma) - v(\sigma) \nabla u(\sigma) u(\sigma) \right) \, d\sigma =
\int_{\partial \Omega} (u(\sigma) \nabla u(\sigma) - v(\sigma) \nabla u(\sigma)) \cdot \hat{N}(\sigma) \, d\sigma =
\int_{\Omega} \nabla \cdot (u \nabla v - v \nabla u) \, dx
\]

\[
= \int_{\Omega} (\nabla u \cdot \nabla v + u \nabla v - \nabla u \cdot \nabla v - u \nabla v) \, dx
\]

\[
= \int_{\Omega} (u \Delta v - v \Delta u) \, dx.
\]

Problem II. Fix \( \epsilon \in (0, 1/2) \). We can choose \( N \) sufficiently large such that for \( |x| > N \), \( 0 < \ln (x^2 + 1) < |x|^{\frac{3\alpha}{2}} \). (For note that for any fixed \( \alpha > 0 \), \( \ln y < y^\alpha \) if \( y \) is sufficiently large. This can be seen e.g. by writing the equivalent statement

\[
y < e^{x^2}
\]

and considering the right-hand side as the expansion \( \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \). Thus \( \ln (x^2 + 1) < \ln (2x^2) < (2x^2)^{\alpha} < x^{3\alpha} \) for \( x \) sufficiently large.) Then for \( |x| > N \),

\[
\frac{\ln (x^2 + 1)}{|x|^{1-\epsilon}} < |x|^{\frac{-3\alpha}{2}}.
\]

So (using the fact that \( 0 \leq \sin^2 x \leq 1 \),

\[
\lim_{M \to \infty} \int_{N}^{M} |f(x)|^2 < \lim_{M \to \infty} \int_{N}^{M} x^{\frac{3\alpha}{2}} = \lim_{M \to \infty} \left( \frac{-1 + 2\epsilon}{2} \right) \left( M^{-\frac{1}{2}+\epsilon} - N^{-\frac{1}{2}+\epsilon} \right)\]
\[
= \left( \frac{1}{2} - \epsilon \right) N^{-\frac{1}{2} + \tau}.
\]

Similarly, since \( f^2 \) is an even function,
\[
\lim_{M \to -\infty} M \int_{-M}^{-N} |f(x)|^2 < \infty.
\]

Okay, now note that \( \lim_{x \to 0} \frac{\ln(x^2 + 1)}{x^2} = 1 \) (e.g., by using the Taylor expansion of \( \ln(x + 1) \) about \( x = 0 \)). So
\[
\lim_{x \to 0} \frac{\ln(x^2 + 1)}{|x|^{1+\epsilon}} = \lim_{x \to 0} \frac{\ln(x^2 + 1)}{x^2} |x|^{1+\epsilon} = \lim_{x \to 0} |x|^{1+\epsilon} = 0.
\]

So \( \lim_{x \to 0} f(x) = 0 \), and thus \( f \) is bounded on the interval \([-N, N]\) (since it's clearly continuous away from 0). It follows that
\[
\int_R f(x)^2 = \int_{-\infty}^{-N} f(x)^2 + \int_{-N}^{N} f(x)^2 + \int_N^{\infty} f(x)^2 < \infty,
\]

since the three terms on the right are finite. So \( f \in L^2(R) \).

**Problem III.** If \( w = 0 \), then \( \langle v, w \rangle = 0 \) by linearity, \( \langle w, v \rangle = 0 \) and the inequality is clear. Now assume that \( w \) is nonzero. Consider the expression
\[
\langle v + tw, v + tw \rangle = t^2 \langle w, w \rangle + 2t \langle v, w \rangle + \langle v, v \rangle
\]

with \( t \) an arbitrary real parameter. This is a quadratic polynomial in \( t \), and is nonnegative since the left-hand side is a squared norm. So the discriminant
\[
4 \langle v, w \rangle^2 - 4 \langle v, v \rangle \langle w, w \rangle
\]
is less than or equal to zero, i.e.,
\[
\langle v, w \rangle^2 \leq \langle v, v \rangle \langle w, w \rangle.
\]

Both sides are nonnegative, and taking square roots gives Cauchy-Schwarz.

Now to prove the triangle inequality, note that by Cauchy-Schwarz,
\[
\langle v + w, v + w \rangle = \langle v, v \rangle + 2 \langle v, w \rangle + \langle w, w \rangle
\]
\[
\leq \langle v, v \rangle + 2 \langle v, w \rangle + \langle w, w \rangle
\]
\[
\leq ||v||^2 + 2 ||v|| ||w|| + ||w||^2
\]
\[
= (||v|| + ||w||)^2
\]

and hence
\[
||v + w|| = \sqrt{\langle v + w, v + w \rangle} \leq ||v|| + ||w||.
\]
Problem IV. Let's check that \( \langle \cdot, \cdot \rangle \) satisfies the properties of an inner product:

- \( \langle f, g \rangle = \int_\mathbb{R} f(x) g(x) dx = \int_\mathbb{R} g(x) f(x) dx = \langle g, f \rangle \)

- \( \langle af + bg, h \rangle = \int_\mathbb{R} \left( af(x) + bg(x) \right) h(x) dx = a \int_\mathbb{R} f(x) g(x) + b \int_\mathbb{R} f(x) g(x) = a \langle f, g \rangle + b \langle f, g \rangle \) by linearity of the integral.

- Of course if \( f \equiv 0 \) then \( \langle f, f \rangle = \int_\mathbb{R} f(x)^2 dx = 0 \). Now let's assume that \( f \) is continuous. If there is an \( a > 0 \) such that \( f(x)^2 = a \) for some \( x \), then by continuity of \( f \) there is an interval \( I = (x - \epsilon, x + \epsilon) \) such that \( f(x)^2 > a/2 \) on \( I \). Then

\[
\int_\mathbb{R} f(x)^2 = \int_{\mathbb{R} \setminus I} f(x)^2 + \int_{I} f(x)^2 \\
\geq \int_{I} f(x)^2 \\
\geq a \epsilon > 0.
\]

So if \( f \neq 0 \) then \( \langle f, f \rangle > 0 \).

The inequality (0.0.6) now follows immediately from Problem III.

Finally, we showed in Problem II that the function

\[
g(x) = \sin(x) \frac{\ln(x^2 + 1)}{|x|^{1/4}}
\]

lies in \( L^2(\mathbb{R}) \), i.e., we can write \( C = \| g \|_{L^2(\mathbb{R})} \) (here we've chosen \( \epsilon = 1/4 \) in the formulation of Problem II). So by Cauchy-Schwarz, if \( f \in L^2(\mathbb{R}) \) then

\[
\int_\mathbb{R} g(x) f(x) \leq \sqrt{\int g(x)^2} \sqrt{\int f(x)^2} \\
= C \left( \int_\mathbb{R} f(x)^2 \right)^{1/2}
\]

as claimed.