1. The Weak Maximum Principle

We will now study some important properties of solutions to the heat equation \( \partial_t u - D \Delta u = 0 \). For simplicity, we sometimes only study the case of 1 + 1 spacetime dimensions, even though analogous properties are verified in higher dimensions.

**Theorem 1.1 (Weak Maximum Principle).** Let \( \Omega \subset \mathbb{R}^n \) be a domain. Recall that \( Q_T \stackrel{\text{def}}{=} (0, T) \times \Omega \) is a spacetime cylinder and that \( \partial_p Q_T \stackrel{\text{def}}{=} \{0\} \times \Omega \cup (0, T] \times \partial \Omega \) is its corresponding parabolic boundary. Let \( w \in C^{1,2}(Q_T) \cap C(\overline{Q_T}) \) be a solution to the (possibly inhomogeneous) heat equation

\[
(1.0.1) \quad w_t - D \Delta w = f,
\]

where \( f \leq 0 \). Then \( w(t, x) \) obtains its max in the region \( \overline{Q_T} \) on \( \partial_p Q_T \). Thus, if \( w \) is strictly negative on \( \partial_p Q_T \), then \( w \) is strictly negative on \( Q_T \).

**Proof.** For simplicity, we consider only case of 1 + 1 spacetime dimensions. Let \( \epsilon \) be a positive number, and let \( u = w - \epsilon t \). Our goal is to first study \( u \), and then take a limit as \( \epsilon \downarrow 0 \) to extract information about \( w \).

Note that on \( Q_T \) we have \( u \leq w \), that \( w \leq u + \epsilon T \), and that in \( Q_T \) we have

\[
(1.0.2) \quad u_t - Du_{xx} = f - \epsilon < 0.
\]

We claim that the maximum of \( u \) on \( \overline{Q}_{T-\epsilon} \) occurs on \( \partial_p Q_{T-\epsilon} \). To verify the claim, suppose that \( u(t, x) \) has its max at \( (t_0, x_0) \in \overline{Q}_{T-\epsilon} \). We may assume that \( 0 < t_0 \leq T - \epsilon \), since if \( t_0 = 0 \) the claim is obviously true. Under this assumption, we have that \( u \leq w \) and that \( w \leq u + \epsilon T \). Similarly, we may also assume that \( x \in \Omega \), since otherwise we would have \( (t, x) \in \partial_p Q_{T-\epsilon} \), and the claim would be true.

Then from vector calculus, \( u_x(t_0, x_0) \) must be equal to 0. Furthermore, \( u_x(t_0, x_0) \) must also be equal to 0 if \( t_0 < T - \epsilon \), and \( u_x(t_0, x_0) \geq 0 \) if \( t_0 = T - \epsilon \). Now since \( u(t_0, x_0) \) is a maximum value, we can apply Taylor’s remainder theorem in \( x \) to deduce that for \( x \) near \( x_0 \), we have

\[
(1.0.3) \quad u(t_0, x) - u(t_0, x_0) = u_x|_{t_0, x_0} (x - x_0) + u_{xx}|_{t_0, x}^* (x - x_0)^2 \leq 0,
\]

where \( x^* \) is some point in between \( x_0 \) and \( x \). Therefore, \( u_{xx}(t_0, x^*) \leq 0 \), and by taking the limit as \( x \to x_0 \), it follows that \( u_{xx}(t_0, x_0) \leq 0 \). Thus, in any possible case, we have that

\[
(1.0.4) \quad u_t(t_0, x_0) - Du_{xx}(t_0, x_0) \geq 0,
\]
which contradicts (1.0.2).

Using \( u \leq w \) and that fact that \( \partial_pQ_{T-\epsilon} \subset \partial_pQ_T \), we have thus shown that

\[
\max_{Q_{T-\epsilon}} u = \max_{\partial_pQ_{T-\epsilon}} u \leq \max_{\partial_pQ_T} w \leq \max_{\partial_pQ_T} w. 
\]  

(1.0.5)

Using (1.0.5) and \( w \leq u + \epsilon T \), we also have that

\[
\max_{Q_{T-\epsilon}} w \leq \max_{Q_{T-\epsilon}} u + \epsilon T \leq \epsilon T + \max_{\partial_pQ_T} w. 
\]  

(1.0.6)

Now since \( w \) is uniformly continuous on \( \overline{Q}_T \), we have that

\[
\max_{\overline{Q}_T} w \uparrow \max_{\overline{Q}_T} w \quad \text{as} \quad \epsilon \downarrow 0. 
\]  

(1.0.7)

as \( \epsilon \downarrow 0 \). Thus, allowing \( \epsilon \downarrow 0 \) in inequality (1.0.6), we deduce that

\[
\max_{\overline{Q}_T} w = \lim_{\epsilon \downarrow 0} \max_{\overline{Q}_{T-\epsilon}} w \leq \lim_{\epsilon \downarrow 0} (\epsilon T + \max_{\partial_pQ_T} w) = \max_{\partial_pQ_T} w \leq \max_{\overline{Q}_T} w. 
\]  

(1.0.8)

Therefore, all of the inequalities in (1.0.8) can be replaced with equalities, and

\[
\max_{\overline{Q}_T} w = \max_{\partial_pQ_T} w 
\]  

(1.0.9)

as desired. \( \square \)

The following very important corollary shows how to compare two different solutions to the heat equation with possibly different inhomogeneous terms. The proof relies upon the weak maximum principle.

**Corollary 1.0.1 (Comparison Principle and Stability).** Suppose that \( v, w \) are solutions to the heat equations

\[
v_t - Dv_{xx} = f, 
\]  

(1.0.10)

\[
w_t - Dw_{xx} = g. 
\]  

(1.0.11)

Then

1. **(Comparison):** If \( v \geq w \) on \( \partial_pQ_T \) and \( f \geq g \), then \( v \geq w \) on all of \( Q_T \).
2. **(Stability):** \( \max_{\overline{Q}_T} |v - w| \leq \max_{\partial_pQ_T} |v - w| + T \max_{\overline{Q}_T} |f - g| \).

**Proof.** One of the things that makes linear PDEs relatively easy to study is that you can add or subtract solutions: Setting \( u \defeq w - v \), we have

\[
u_t - Du_{xx} = g - f \leq 0. 
\]  

(1.0.12)

Then by Theorem 1.1, since \( u \leq 0 \) on \( \partial_pQ_T \) we have that \( u \leq 0 \) on \( Q_T \). This proves (1).

To prove (2), we define \( M \defeq \max_{\overline{Q}_T} |f - g| \), \( u \defeq w - v - tM \) and note that
Thus, by Theorem 1.1, we have that
\[
\max_{Q_T} u = \max_{\partial Q_T} u \leq \max_{\partial Q_T} |w - v|.
\]

Thus, subtracting and adding \(tM\), we have
\[
\max_{Q_T} w - v \leq \max_{Q_T} (w - v - tM) + \max_{\partial Q_T} tM \leq \max_{\partial Q_T} |w - v| + TM.
\]

Similarly, by setting \(u \overset{\text{def}}{=} v - w - tM\), we can show that
\[
\max_{Q_T} v - w \leq \max_{\partial Q_T} |w - v| + TM.
\]

Combining (1.0.15) and (1.0.16), and recalling the definition of \(M\), we have shown (2). \(\square\)