Part II (50 points)

1. (Thurs., Oct. 26; Second Fundamental Theorem; 3 + 2 + 2 + 3 + 3 + 3 = 16 points) Let $sinc(x)$ denote the "sinc" function

$$sinc(x) = \begin{cases} 
1 & \text{if } x = 0, \\
\frac{\sin x}{x} & \text{if } x \neq 0.
\end{cases}$$

Now consider the "sine integral" function

$$Si(x) = \int_0^x sinc(t) \, dt.$$ 

Both of these functions frequently come up in Fourier analysis and signal processing and hence have been given their own names. Remark: $Si(x)$ cannot be expressed in terms of standard elementary functions.

a) Compute $Si'(x)$ and $Si''(x)$. You will have to compute $Si''(0)$ by using the definition of the derivative. Hint: In computing $Si''(0)$, you can make use of the fact that $\sin(\Delta x) = \Delta x + O((\Delta x)^3)$.

Solution: By the second fundamental theorem of calculus, $Si'(x) = sinc(x)$. Hence, $Si''(x) = sinc'(x)$.

For $x \neq 0$, we can use the quotient rule to get

$$sinc'(x) = \frac{x \cos x - \sin x}{x^2}.$$ 

For $x = 0$, we need to use the analytic definition of the derivative, which says that

$$sinc'(0) = \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} - 1 = \lim_{\Delta x \to 0} \frac{1 - 1 + O((\Delta x)^2)}{\Delta x} = \lim_{\Delta x \to 0} O(\Delta x) = 0.$$ 

b) List the critical points of $Si(x)$ in the entire range $-\infty < x < \infty$. Which critical points are local maxima and which ones are local minima?

Solution: The critical points are where $sinc(x) = Si'(x) = 0$. This happens when $\sin(x) = 0$, i.e., at $x = n\pi$ for $n \in \mathbb{Z}$ not equal to 0, since $sinc(0) = 1$. To figure out which ones are local maxima/minima, we look at the sign of $sinc(x)$ near $n\pi$. First note that since the function is even, if $n\pi$ is a local max/min, so is $-n\pi$. So we can assume $n$ is positive, i.e., $x$ is positive. For $n$ even, and $x$ slightly less than $n\pi$, $\sin(x) > 0$, and hence so is $sinc(x)$, and for $x$ slightly greater than $n\pi$, $sinc(x) < 0$. The situation is reversed for $n$ odd. Hence, the local maxima occur at $x = n\pi$ for $n$ even and nonzero and the local minima occur at $n\pi$ for $n$ odd.
c) Draw a rough sketch of $S_i'(x)$ and $S_i''(x)$. The drawings only have to be qualitatively correct, but make sure that the zeros of $S_i'(x)$ are accurately displayed.

**Solution:**

Graph of $S_i'(x)$:

The zeros of $S_i'(x)$ are at $x = n\pi$ for $n \in \mathbb{Z}$. 
Graph of $S_i''(x)$:

The zeros of $S_i''(x)$ are implicitly given by $x = \tan(x)$. 
d) Sketch the graph of \( \text{Si}(x) \) on the interval \(-10\pi \leq x \leq 10\pi\) with labels for the critical points and inflection points. The drawing should be qualitatively correct and should reflect the shape of the graphs you sketched in part c).

**Solution:**
Zoomed in graph for \(-4\pi < x < 4\pi\):

The graph has only one zero at \(x = 0\) and is the graph of an odd function. The blue points are the critical points at \(x = n\pi\). The green lines are the approximate solutions to \(x = \tan x\) and mark the inflection points. The graph continues in this pattern all the way out to \(10\pi\). It looks like a wave whose amplitude is getting smaller.
e) Let $r > 1$ be a real number, and define

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{\sin(x^r)}{x} & \text{if } x \neq 0. \end{cases}$$

Remark: It is not too hard to show that $f(x)$ is continuous, even at $x = 0$. Consider the function

$$h(x) = \int_0^x f(t) \, dt.$$ 

Show that $h(x)$ can be expressed in terms of composition of $\text{Si}$ with another function.

**Solution:** Consider the function $F(x) = \frac{\text{Si}(x^r)}{r}$. By the chain rule,

$$F'(x) = \text{Si}'(x^r) \frac{d}{dx}(x^r) = \text{sinc}(x^r)x^{r-1} = f(x).$$

Hence, by the first fundamental theorem of calculus

$$h(x) = F(x) - F(0) = F(x).$$

Thus, $h(x)$ is, up to a scaling factor, the composition of $\text{Si}(x)$ with $x^r$.

f) Compute

$$\lim_{x \to 3} \frac{x^2}{x - 3} \int_3^x \text{sinc}(t) \, dt.$$ 

**Solution:** By the fundamental theorem of calculus, the limit is equal to

$$\lim_{x \to 3} \frac{x^2}{x - 3} (\text{Si}(x) - \text{Si}(3)) = 9 \lim_{x \to 3} \frac{\text{Si}(x) - \text{Si}(3)}{x - 3} = 9 \lim_{\Delta x \to 0} \frac{\text{Si}(3 + \Delta x) - \text{Si}(3)}{\Delta x} = 9 \text{Si}'(3) = 3 \sin 3.$$
2. (Fri., Oct. 27; volumes by slicing; $4 + 1 = 5$ points) 7.3: 22

Solution:

(a) If $V$ is the volume in the bowl and $A(h)$ is the surface area with the liquid having total height $h$, then

$$\frac{dV}{dt} = -cA(h)$$

for some constant $c > 0$. By the disks method,

$$V = \int_{0}^{h} A(x) \, dx.$$ 

Hence, by the fundamental theorem of calculus,

$$\frac{dV}{dh} = A(h).$$

Using the chain rule, we have

$$-cA(h) = \frac{dV}{dh} \frac{dh}{dt} = A(h) \frac{dh}{dt}.$$ 

Hence, $\frac{dh}{dt} = -c$.

(b) By integrating both sides of the equation obtained from part (a), we get

$$\int_{0}^{t} dh = \int_{0}^{t} -c \, dt$$

and hence

$$h(t) - h(0) = -ct.$$ 

Thus, if $h(t) = 0$, then $t = \frac{h_0}{c}$.

3. (Fri., Oct. 27; volumes by slicing; 10 points) Find the volume of the three-dimensional solid with $x > 0, y > 0, z > 0$ and $z^4 < x + y < z$.

Hint: First find the area of the horizontal cross sections, which are perpendicular to the $z$ axis.

Solution: First note that the $z$ values go from 0 to 1, as $z^4 < z$. Now, the horizontal cross sections at a fixed height $0 < z < 1$ is the parallelogram bounded by the $x$-axis, the $y$-axis and the lines $x + y = z^4, x + y = z$. Hence, the area of the cross section is the difference between the area of the big triangle formed by the two axes and the line $x + y = z$ and the area of the smaller triangle formed by the axes and the line $x + y = z^4$. The big triangle has base and height $z$ and hence area $\frac{z^2}{2}$, while the small triangle has area $\frac{z^6}{2}$. Hence, the area of the cross section is $A(z) = \frac{z^2 - z^8}{2}$.
The volume of the solid is
\[ \int_0^1 A(z) \, dz = \int_0^1 \frac{z^2 - z^8}{2} \, dz = \frac{1}{6} - \frac{1}{18} = \frac{1}{9}. \]

4. (Tues., Oct. 31; shell and disk method; 2 + 1 + 2 + 1 = 6 points) (Donut with triangular cross sections)

a) An equilateral triangle in the \((x, y)\) plane of side length \(\ell\) has a base that runs along the \(x\) axis. The center of the triangle is a distance \(R\) from the \(y\) axis, where \(R > \frac{1}{2} \ell\) (and thus the triangle does not intersect the \(y\) axis). The triangle is revolved around the \(y\) axis to create a solid. Use the cylindrical shell method to express the volume of the solid in terms of an integral. Your answer should depend on \(\ell\) and \(R\).

**Solution:** The triangle is built up of two parts. Between \(x = R - \frac{\ell}{2}\) and \(x = R\), the triangle is the region under the graph
\[ y = \tan \left( \frac{\pi}{3} \right) \left( x - R + \frac{\ell}{2} \right) = \sqrt{3} \left( x - R + \frac{\ell}{2} \right). \]

From \(x = R\) to \(x = R + \frac{\ell}{2}\), the triangle is the region under the graph
\[ y = -\sqrt{3} \left( x - R - \frac{\ell}{2} \right). \]

You can obtain these formulas by computing the slope and the \(x\)-intercept of the corresponding lines and noting that equilateral triangles have \(\frac{\pi}{3}\) as all their angles. Hence, using the shells method on each piece separately, we get that
\[ V = 2\pi \sqrt{3} \int_{R-\frac{\ell}{2}}^{R} x \left( x - R + \frac{\ell}{2} \right) \, dx + 2\pi \sqrt{3} \int_{R}^{R+\frac{\ell}{2}} x \left( R + \frac{\ell}{2} - x \right) \, dx. \]

b) Compute the integral from part a) to find a formula for the volume.

**Solution:** One possible way to solve the problem is just to do a straightforward integration but this leads to messy algebra. Let us simplify the problem a little. First, we split the integral into two parts:

\[ V_1 = 2\pi \sqrt{3} \left( \int_{R-\frac{\ell}{2}}^{R} \frac{x\ell}{2} \, dx + \int_{R}^{R+\frac{\ell}{2}} \frac{x\ell}{2} \, dx \right) \]

and
\[ V_2 = 2\pi \sqrt{3} \left( \int_{R-\frac{\ell}{2}}^{R} x(x - R) \, dx + \int_{R}^{R+\frac{\ell}{2}} x(R - x) \, dx \right). \]

The sums in the first integral can be combined since we are integrating the same function and the end points much up. Hence,
\[
V_1 = 2\pi\sqrt{3} \int_{R-l}^{R+l} \frac{x}{2} \, dx = \pi\sqrt{3}l \left( R + \frac{l}{2} \right)^2 - \left( R - \frac{l}{2} \right)^2 = \pi\sqrt{3}Rl^2.
\]

To solve for \( V_2 \), we make the substitution \( u = x - R \). Then,

\[
V_2 = 2\pi\sqrt{3} \left( \int_{-\frac{l}{2}}^{0} u^2 + Ru \, du + \int_{0}^{\frac{l}{2}} -u^2 - Ru \, du \right) \, du
= 2\pi\sqrt{3} \left( 0 + \frac{l^3}{24} - \frac{Rl^2}{8} - \frac{l^3}{24} - \frac{Rl^2}{8} \right)
= -\frac{\sqrt{3}\pi Rl^2}{2}
\]

Adding everything together, we get

\[
V = \frac{\sqrt{3}\pi Rl^2}{2}.
\]

c) Repeat parts a) and b), but this time using the disk method.

**Solution:** To use the disks method, we need to switch the roles of \( x \) and \( y \) in the formula given in the book, because we are revolving around the \( y \)-axis. Since the solid is separated from the \( y \)-axis, the volume is given by the formula

\[
V = \int \pi(x_1^2 - x_2^2) \, dy.
\]

where \( x_1(y) \) is the function that describes the line further away from the \( y \)-axis, and \( x_2(y) \) is the function describing the line closer to the \( y \)-axis. By the same method as in part (a), we can find the equation of the lines

\[
x_1 = R + \frac{l}{2} - \frac{y}{\sqrt{3}} \text{ and } x_2 = \frac{y}{\sqrt{3}} + R - \frac{l}{2}.
\]

The bounds of the integral are given by the range of the \( y \)-values. The minimum value is 0 and the maximum is the height of the triangle, which is \( \frac{\sqrt{3}l}{2} \). Hence, the volume is

\[
V = \pi \int_{0}^{\frac{\sqrt{3}l}{2}} \frac{y^2}{3} - 2\frac{y}{\sqrt{3}} \left( R + \frac{l}{2} \right)^2 - \frac{y^2}{3} - 2\frac{y}{\sqrt{3}} \left( R - \frac{l}{2} \right)^2 \, dy
= \pi \int_{0}^{\frac{\sqrt{3}l}{2}} -4\frac{y}{\sqrt{3}} R + 2Rl \, dy
= \pi R \left( l^2 \sqrt{3} - \frac{\sqrt{3}l^2}{2} \right) = \frac{\sqrt{3}\pi Rl^2}{2}.
\]
5. (Tues., Oct. 31; shell method; 3 + 7 = 10 points) 7.4: 12, 13.

**Remark:** Think of the “spherical ring” as a sphere that has been gored by a cylinder whose radius is smaller than the radius of the sphere, but whose length is infinite.

**Solution:**

12. The solid is a cone with vertex at \((0, h)\) and bottom face at \(y = 0\), with radius equal to \(r\). So the formula we expect to get is \(V = \frac{\pi r^2 h}{3}\). Let us derive this using the cylindrical shell method.

\[
V = 2\pi \int_0^r xy \, dx = 2\pi h \int_0^r x \left(1 - \frac{x}{r}\right) \, dx = 2\pi h \left(\frac{x^2}{2} \bigg|_0^r - \frac{x^3}{3r} \bigg|_0^r\right) = 2\pi h \left(\frac{r^2}{2} - \frac{r^3}{3}\right) = \frac{\pi r^2 h}{3}
\]
as expected.

13. We can obtain the spherical ring by taking a segment of a circle of radius \(a\) and length of base \(2h\) (with the base parallel to the \(y\)-axis) and revolving it around the \(y\)-axis. Since the base is parallel to the \(y\)-axis, it’s equation is just \(x = r\) for some constant \(r\). To figure out this constant, note that \(r^2 + h^2 = a^2\). Hence, \(r = \sqrt{a^2 - h^2}\). By symmetry the surface above and below the \(x\)-axis will contribute the same total volume. Hence, we simply compute the volume of the solid obtained by rotating the piece of the ring above the \(x\)-axis and then multiply by 2.

The bounds on the \(x\)-value of the region that is revolved are \(r\) and \(a\) and the equation for the graph bounds the region is \(y = \sqrt{a^2 - x^2}\). Hence, by the shells method,

\[
V = 4\pi \int_r^a x \sqrt{a^2 - x^2} \, dx.
\]

We use the substitution \(u = a^2 - x^2\). Then, \(du = -2x \, dx\). Hence,

\[
V = -2\pi \int_{a^2-r^2}^{a^2} \sqrt{u} \, du = 2\pi \int_0^{a^2-r^2} \sqrt{u} \, du = 2\pi \int_0^{h^2} \sqrt{u} \, du = 2\pi \left[\frac{2u^{\frac{3}{2}}}{3}\right]_0^{h^2} = \frac{4\pi h^3}{3}.
\]