Problem 1.

a) We find,

\[
v'(9) \approx \frac{v(9) - v(8)}{1} = \frac{90.744 - 79.931}{1} = 10.813
\]

\[
v'(10) \approx \frac{v(10) - v(9)}{1} = \frac{101.983 - 90.744}{1} = 11.239
\]

b) We find,

\[
v''(10) \approx \frac{v'(10) - v'(9)}{1} = \frac{11.239 - 10.813}{1} = 0.426
\]

c) A quadratic approximation of \( v(d) \) near \( d = 10 \) is,

\[
v(d) \approx v(10) + v'(10)(d - 10) + \frac{v''(10)}{2}(d - 10)^2
\]

Using our approximations of \( v'(10) \) and \( v''(10) \) from parts a) and b) gives,

\[
v(d) \approx 101.983 + 11.239(d - 10) + \frac{0.426}{2}(d - 10)^2
\]

Using equation (1), we can now estimate the value \( v(11) \):

\[
v(11) \approx 101.983 + 11.239 + \frac{0.426}{2} = 113.435
\]

Problem 2.

a) To show that \( f(E) \) is continuous at \( E = 0 \) we must show that,

\[
\lim_{E \to 0^+} \frac{\tanh^{-1}(E)}{E} = f(0) = 1
\]

Note that we are only considering the right-handed limit as \( E \geq 0 \). We proceed by evaluating the limit:

\[
\lim_{E \to 0^+} \frac{\tanh^{-1}(E)}{E} = \lim_{E \to 0^+} \frac{\tanh^{-1}(E) - \tanh^{-1}(0)}{E - 0}
\]

\[
= \frac{d}{dE} \tanh^{-1}(E) \bigg|_{E=0}
\]

\[
= \frac{1}{1 - E^2} \bigg|_{E=0}
\]

\[
= 1
\]

We have shown that equation (2) is true and so \( f(E) \) is continuous at \( E = 0 \).
b) **Method 1: Implicit Differentiation**

\[ y = \tanh^{-1}(E) \]
\[ \tanh(y) = E \]

Taking the derivative of both sides gives,

\[ \frac{d}{dE} \tanh(y) = \frac{d}{dE} (E) \]
\[ \text{sech}^2(y) \frac{dy}{dE} = 1 \]
\[ \frac{dy}{dE} = \frac{1}{\text{sech}^2(y)} \]

Using the identity \( \text{sech}^2(y) = 1 - \tanh^2(y) \) we find,

\[ \frac{dy}{dE} = \frac{1}{1 - \tanh^2(y)} \]
\[ = \frac{1}{1 - E^2} \]

**Method 2: Explicit Differentiation**

First, we find an explicit expression for \( y = \tanh^{-1}(x) \):

\[ \tanh(y) = E \]
\[ \frac{e^y - e^{-y}}{e^y + e^{-y}} = E \]
\[ (1 - E)e^y = (1 + E)e^{-y} \]
\[ e^{2y} = \frac{1 + E}{1 - E} \]
\[ e^y = \sqrt{\frac{1 + E}{1 - E}} \]
\[ y = \frac{1}{2} \ln \left( \frac{1 + E}{1 - E} \right) \]  

(3)

We now differentiate equation (3):

\[ \frac{dy}{dE} = \frac{1}{2} \left( \frac{1 + E}{1 - E} \right) \left( \frac{1 - E + 1 + E}{(1 - E)^2} \right) \]
\[ = \frac{1 - E}{1 + E} \left( \frac{1}{(1 - E)^2} \right) \]
\[ = \frac{1}{1 - E^2} \]

c) We know that \((1 + x)^r \approx 1 + rx\). In this case, we have \(x = -E^2\) and \(r = -1\) and so,

\[ \frac{1}{1 - E^2} \approx 1 + E^2 \]
d) We are told to assume that,
\[ \frac{d}{dE} (B_0 + B_1 E + B_2 E^2 + B_3 E^3) = 1 + E^2 \]
\[ B_1 + 2B_2 E + 3B_3 E^2 = 1 + E^2 \]
From this, we find that,
\[ B_1 = 1, \quad B_2 = 0, \quad B_3 = \frac{1}{3} \]
We also know that \( B_0 = \text{tanh}^{-1}(0) = 0 \). So, our cubic approximation is,
\[ \text{tanh}^{-1}(E) \approx E + \frac{E^3}{3} \] (4)

e) Using equation (4), we find that,
\[ f(E) = \frac{\text{tanh}^{-1}(E)}{E} \approx E + \frac{E^3}{3} = 1 + \frac{E^2}{3} \] (5)

f) Using equation (5), we find that,
\[ S(E) = 2\pi \left( 1 + f(E) (1 - E^2) \right) \]
\[ \approx 2\pi \left( 1 + \left(1 + \frac{E^3}{3}\right) (1 - E^2) \right) \]
\[ \approx 2\pi \left( 1 + 1 - E^2 + \frac{E^2}{3} \right) \]
\[ = 4\pi \left( 1 - \frac{E^2}{3} \right) \] (6)

f) From equation (6), we see that for \( E > 0 \), \( S(E) < S(0) \). Therefore, slightly squashing the sphere causes the surface area to decrease.

Problem 3.

18.
First, we determine the critical points of \( y(x) \) by examining the derivative,
\[ y'(x) = mx^{m-1} (1-x)^n - nx^m (1-x)^{n-1} \]
\[ = x^{m-1} (1-x)^{n-1} [m (1-x) - nx] \]
\[ = (m+n) x^{m-1} (1-x)^{n-1} \left[ \frac{m}{m+n} - x \right] \]
We see that there are three critical points at,
\[ x = 0, \frac{m}{m+n}, 1 \]
a) Consider the sign of \( y'(x) \) around \( x = 0 \):

<table>
<thead>
<tr>
<th>( x^{m-1} ) (odd function)</th>
<th>( m+n )</th>
<th>( m+n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; 0 )</td>
<td>0 &lt; ( x ) &lt; ( \frac{m}{m+n} )</td>
<td></td>
</tr>
<tr>
<td>( (1-x)^{n-1} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( m+n - x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y'(x) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
We see that \( x = 0 \) is a minimum if \( m \) is even.
b) Consider the sign of \( y'(x) \) around \( x = 1 \):

<table>
<thead>
<tr>
<th></th>
<th>( m/n &lt; x &lt; 1 )</th>
<th>( 1 &lt; x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^{m-1} )</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( (1-x)^{n-1} ) (odd function)</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( \frac{m}{m+n} - x )</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( y'(x) )</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

We see that \( x = 1 \) is a minimum if \( n \) is even.

c) Consider the sign of \( y'(x) \) around \( x = \frac{m}{m+n} \):

<table>
<thead>
<tr>
<th></th>
<th>( 0 &lt; x &lt; \frac{m}{m+n} )</th>
<th>( \frac{m}{m+n} &lt; x &lt; 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^{m-1} )</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( (1-x)^{n-1} )</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( \frac{m}{m+n} - x )</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( y'(x) )</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

We see that \( x = \frac{m}{m+n} \) is always a maximum.

22.

In order for \( f(x) \) to have critical points at \( x = -2 \) and \( x = 1 \), \( f'(x) \) must have zeros at \( x = -2 \) and \( x = 1 \). Further, in order for \( x = -2 \) to be a maximum and \( x = 1 \) to be a minimum, we need \( f'(x) > 0 \) for \( x < -2 \), \( f'(x) < 0 \) for \( -2 < x < 1 \), and \( f'(x) > 0 \) for \( x > 1 \). We can satisfy these requirements by choosing,

\[
f'(x) = (x + 2)(x - 1) = x^2 + x - 2
\]

By inspection, we find that the function \( f(x) \) that corresponds to this derivative is,

\[
f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x
\]

Problem 4.

12.

To identify points of inflection, we need to consider the second derivative of \( y = \frac{12}{x^2} - \frac{12}{x} \).

\[
y'(x) = 12 \left( -\frac{2}{x^3} + \frac{1}{x^2} \right)
\]

\[
y''(x) = 12 \left( \frac{6}{x^4} - \frac{2}{x^3} \right)
= \frac{24(3-x)}{x^4}
\]

We see that \( y''(x) \) has a single zero at \( x = 3 \) and that \( y''(x) > 0 \) for \( x < 3 \) and \( y''(x) < 0 \) for \( x > 3 \). Therefore, there is a single point of inflection at \( x = 3 \) and the graph is concave up for \( x < 3 \) and concave down for \( x > 3 \).
It is not possible. Assume that $f'(x) < 0$ and $f''(x) < 0$ for all $x$. This means that for two points $x_2 > x_1$, $f'(x_2) < f'(x_1) < 0$ (i.e. the slope of the graph gets more and more negative as $x$ increases). Therefore, since the function $f$ decreases at a faster and faster rate as $x$ increases, if $f(x_0) > 0$ for some $x_0$, the graph $y = f(x)$ must eventually cross the $x$-axis at some $x > x_0$.

Using implicit differentiation we find,

\[
\frac{d^2}{dx^2} (x^2 + y^2) = \frac{d^2}{dx^2} (a^2) \\
\frac{d}{dx} \left(2x + 2y \frac{dy}{dx} \right) = 0 \\
1 + \frac{dy}{dx} \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0
\]

(7)

As the numerator of the RHS of equation (8) is always positive, the sign of $y''$ is always opposite to the sign of $y$.

Finally, we can re-express equation (8) in terms of $x$. From equation (7), we see that $\frac{dy}{dx} = -\frac{x}{y}$. Substituting this into equation (8) gives,

\[
\frac{d^2y}{dx^2} = -\frac{1 + \frac{x^2}{y^2}}{y} = -\frac{x^2 + y^2}{y^3}
\]

As $x^2 + y^2 = a^2$ is a multi-valued function, we have $y = \pm \sqrt{a^2 - x^2}$, and so,

\[
\frac{d^2y}{dx^2} = \begin{cases} 
\frac{-a^2}{(a^2-x^2)^{3/2}} & \text{for top half of circle } (y > 0) \\
\frac{a^2}{(a^2-x^2)^{3/2}} & \text{for bottom half of circle } (y < 0)
\end{cases}
\]
24.

To identify points of inflection, we need to consider the second derivative of $y$:

\[
\begin{align*}
    y'(x) &= 3ax^2 + 2bx + c \\
    y''(x) &= 6ax + 2b
\end{align*}
\]

We see that there is a single inflection point at,

\[
    0 = 6ax + 2b \\
    x = -\frac{b}{3a}
\]

Next, we consider the critical points where $y'(x) = 0$:

\[
\begin{align*}
    3ax^2 + 2bx + c &= 0 \\
    x &= \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} \\
    &= \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}
\end{align*}
\]

There are three cases to consider:

1. $b^2 > 3ac$: $y(x)$ has one critical point to the left of the inflection point and one critical point to the right of the inflection point.
2. $b^2 = 3ac$: $y(x)$ has a single critical point at the point of inflection.
3. $b^2 < 3ac$: $y(x)$ no critical points.

An example of each case is shown below.

![Graphs showing inflection and critical points for different cases](image-url)

**Figure 2**