Linear + Quadratic Approximations

- Linear approximation
  
  \[
  y = f(x_0) + f'(x_0)(x - x_0)
  \]

- The tangent line approximates \( f(x) \).
  It gives a good approximation near \( x_0 \):
  \[
  f(x) \approx f(x_0) + f'(x_0)(x-x_0)
  \]
  (when \( x \approx x_0 \))

- The approximation might be very bad when \( x \) is not near \( x_0 \).

  Alternate notation: \( f(x) = f(x_0) + f'(x_0) \Delta x + O((\Delta x)^2) \), \( \Delta x = x - x_0 \)

- Ex: \( f(x) = \ln x \), \( x_0 = 1 \) (base point)
  \[
  f'(x) = \frac{1}{x}, \quad f(1) = 0, \quad f'(1) = 1.
  \]

  \( \ln x \approx f(1) + f'(1)(x-1) = 0 + 1 \cdot (x-1) = x-1 \) when \( x \) is near 1.
Building block list of linear approximations:
(we assume $x_0 = 0$ is the basepoint and $|x| < \epsilon$)

1. $\sin(x) \approx x$ when $x \approx 0$
2. $\cos(x) \approx 1$ when $x \approx 0$
3. $e^x \approx 1 + x$ when $x \approx 0$
4. $\ln(1+x) \approx x$ when $x \approx 0$
5. $(1+x)^r \approx 1 + rx$ when $x \approx 0$.

You should learn how to quickly derive these approximations.

Proof of (1):

If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.

$f(0) = 0$, $f'(0) = 1$
Therefore, $\sin(x) \approx 0 + 1 \cdot (x-0) = x$ when $x \approx 0$.

The proofs of 2-5 are similar. We already proved (4).

Proof of (5): $f(x) = (1+x)^r$, $f'(x) = r (1+x)^{r-1}$

$f(0) = 1$, $f'(0) = r$
Therefore, $f(x) = (1+x)^r \approx 1 + r(x-0) = 1 + rx$ when $x \approx 0$. 
Ex: Find the linear approximation of \( f(x) = \frac{e^{-2x}}{\sqrt{1+x}} \) near \( x_0 = 0 \).

- We can use the building blocks to give a short solution (without calculating \( f'(x) \)).

\[ e^{-2x} \approx 1 + (-2x) = 1 - 2x \]

\[ \frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} \approx 1 - \frac{1}{2} x. \]

\[ \frac{e^{-2x}}{\sqrt{1+x}} \approx (1-2x)(1-\frac{1}{2}x) \approx 1 - \frac{5}{2} x \text{ when } x \approx 0. \]

- Note that we have ignored all \( x^2, x^3 \) etc. terms. When \( x \approx 0 \), these terms are very small compared to \( 1 - \frac{5}{2}x \).

- Note that \( f(x) \approx 1 - \frac{5}{2}x \) means that

\[ f'(x) = \frac{5}{2} \] (we didn't even have to compute a formula for \( f'(x) \) !)

Ex: Compute \( \lim_{x \to 0} \frac{(1+2x)^{10} - 1}{x} \). Use \( (1+2x)^{10} \approx 1 + (10)(2x) = 1 + 20x \)

- \( \lim_{x \to 0} \frac{1}{x} \frac{C(1+2x)^{10} - 1}{x} = \lim_{x \to 0} \frac{x+20x}{x} = 20 \).
Quadratic Approximations

- Often times linear approximations are not accurate enough.

Here is the basic formula for quadratic approximations:

\[
f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2}
\]

when \( x \approx x_0 \)

- Note: If \( f(x) = Ax^2 + Bx + C \), then
  \[
  f'(x) = 2Ax + B
  \]
  \[
  f''(x) = 2A
  \]
  Thus, for \( x_0 = 0 \), \( f(0) = C \), \( f'(0) = B \), \( f''(0) = 2A \),
  and the quadratic approximation to \( f(x) \)
  is \( f(x) \approx g + B \cdot x + \frac{1}{2} \cdot 2A \cdot x^2 \)
  \[
  = Ax^2 + Bx + C
  \]
  This explains the \( \frac{1}{2} \) in the formula and shows that the quadratic approximation is exact when \( f \) is a degree 2 polynomial.
\[ f(x) = \cos x \quad f(0) = 1 \]
\[ f'(x) = -\sin x \quad f'(0) = 0 \]
\[ f''(x) = -\cos x \quad f''(0) = -1 \]

\[
 f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2
\]

\[
 = 1 + 0 \cdot x + \frac{1}{2} \cdot (-1) \cdot x^2
\]

\[
 = 1 - \frac{1}{2} x^2 \quad \text{when } x \approx 0.
\]

**Building block quadratic approximations**

1. \( \sin(x) \approx x \quad \text{for } x \approx 0, \ XXO \)
2. \( \cos(x) \approx 1 - \frac{1}{2} x^2 \quad \text{for } x \approx 0 \)
3. \( e^x \approx 1 + x + \frac{1}{2} x^2 \quad \text{for } x \approx 0 \)
4. \( \ln(1 + x) \approx x - \frac{1}{2} x^2 \quad \text{for } x \approx 0 \)
5. \( (1 + x)^r \approx 1 + rx + \frac{r(r-1)}{2} x^2 \quad \text{for } x \approx 0 \)

**Proofs:** Are not that interesting.

Just compute \( f(x) \), \( f'(x) \), \( f''(x) \).
Ex: Find the quadratic approximation to \( f(x) = \frac{e^{2x}}{\sqrt{1 + x}} \) near \( x = 0 \)

We can use the quadratic approximation building blocks to give a relatively short answer:

\[
e^{2x} \approx 1 + (-2x) + \frac{1}{2} (-2x)^2 = 1 - 2x + 2x^2
\]

\[
\frac{1}{\sqrt{1 + x}} \approx (1 + x)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)(-\frac{1}{2} - 1)}{2} x^2
\]

\[
= 1 - \frac{1}{2}x + \frac{3}{8}x^2.
\]

\[
f(x) \approx \left(1 - 2x + 2x^2\right) \left(1 - \frac{1}{2}x + \frac{3}{8}x^2\right)
\]

\[
\approx 1 - \frac{5}{2}x + \frac{27}{8}x^2 \text{ when } x \to 0.
\]

We have ignored all cubic and higher order terms since we are "expanding only to quadratic order".