* Taylor Series

- Recall the geometric series:
  
  \[ 1 + x + x^2 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1 \]

- A general power series is an infinite sum
  
  \[ f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \]

  that represents a function \( f(x) \) when \( |x| < R \).

- \( R \) is called the radius of convergence. In particular,
  
  when \( |x| < R \), \( |a_n x^n| \to 0 \) as \( n \to \infty \). On the other hand, for \( |x| > R \), \( |a_n x^n| \) does not go to 0 as \( n \to \infty \).

- For example, in the case of the geometric series, all of the \( a_n \) are equal to 1. Thus, if \( x = \frac{1}{2} \), then \( |a_n x^n| = \frac{1}{2^n} \). The higher order terms therefore become increasingly negligible as \( n \to \infty \) when \( x = \frac{1}{2} \).
Ex. If $|x| = 1$, then in the geometric series, $|a_n x^n| = 1$ does not lead to 0. The infinite sum $1 - 1 + 1 - 1 + \ldots$ bounces back and forth between 0 and 1.

- When $|x| > 1$, the geometric series diverges.

- **Basic tools**
  
  Rules of polynomials apply within the radius of convergence.

  Since $\frac{1}{1-x} = 1 + x + x^2 + \ldots$.

  **Ex:** Substitution $x = -u$

  $\frac{1}{1+u} = 1 - u + u^2 - u^3 + \ldots$

  **Ex:** Substitution $x = -v^2$

  $\frac{1}{1+v^2} = 1 - v^2 + v^4 - v^6 + \ldots$
Ex: Term by term multiplication

\[
(\frac{1}{1-x}) (\frac{1}{1-x}) = (1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots)
\]

\[
= 1 + 2x + 3x^2 + \ldots
\]

- Remember that \( x \) here is some number like \( \frac{1}{2} \). As you take higher and higher powers of \( x \), the terms get smaller and smaller.

Ex: Term by term differentiation

\[
\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx} \left[ 1 + x + x^2 + x^3 + \ldots \right]
\]

\[
= 1 + 2x + 3x^2 + \ldots
\]

(AGrees with previous example)

Ex: Term by term integration

\[
\int \frac{du}{1+u} = \int (1-u+u^2-u^3+\ldots)\,du = c + u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \ldots
\]

\[
\ln (1+x) = \int_{0}^{x} \frac{du}{1+u} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots
\]
Ex
\[
\int \frac{dv}{1 + v^2} = \int \left(1 - v^2 + v^4 - v^6 + \ldots\right)dv
= \frac{v}{1} + v - \frac{v^3}{3} + \frac{v^5}{5} - \frac{v^7}{7} + \ldots
\]

\[\tan^{-1} x = \int_0^x \frac{dv}{1 + v^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\]
Taylor's Series and Taylor's Formula

If \( f(x) = a_0 + a_1 x + a_2 x^2 + \ldots \), we want to figure out what all of the coefficients are.

Differentiating term by term, we have:

- \( f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots \)
- \( f''(x) = (2)(1) a_2 + (3)(2) a_3 x + (4)(3) a_4 x^2 + \ldots \)
- \( f'''(x) = (3)(2)(1) a_3 + (4)(3)(2) a_4 x + \ldots \)

Plugging in \( x = 0 \), we have:

- \( f(0) = a_0 \)
- \( f'(0) = a_1 \)
- \( f''(0) = 2a_2 \)
- \( f'''(0) = 3! a_3 \)

In general, Taylor's formula holds:

\[
f^{(n)}(0) = n! \cdot a_n, \quad a_n = \frac{1}{n!} f^{(n)}(0)
\]
\[ f(x) = e^x \]
\[ f'(x) = e^x \]
\[ f''(x) = e^x \]
\[ \vdots \]
\[ f^{(n)}(x) = e^x \]

\[ \Rightarrow f^{(n)}(0) = 1 \]

By Taylor's formula,
\[ a_n = \frac{1}{n!} \]

Hence,
\[ e^x = \frac{1}{0!} + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots \]

In a more compact form:
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

Now we can calculate \( e \) to any desired degree of accuracy:
\[ e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots \]
Ex: \[ f(x) = \cos x \]
\[ f'(x) = -\sin x \]
\[ f''(x) = -\cos x \]
\[ f'''(x) = \sin x \]
\[ f^{(4)}(x) = \cos x \]
\[ f^{(5)}(x) = -\sin x \]
\[ f^{(6)}(x) = -\cos x \]
\[ f^{(7)}(x) = \sin x \]
\[ f^{(8)}(x) = 0. \]

Only even coefficients are non-zero and their signs alternate. Therefore,
\[
\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6
\]
To find the Taylor series for $\sin x$, we can either proceed as in the case of $\cos x$, or alternatively, just differentiate the series for $\cos x$:

$$\sin x = \frac{d}{dx} \cos x = 0 - 2 \cdot \frac{1}{2!} x + 4 \cdot \frac{1}{4!} x^3 - 6 \cdot \frac{1}{6!} x^5 + \cdots$$

$$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Let's compare these Taylor series with the quadratic approximation from earlier in the semester:

$$\cos x \approx 1 - \frac{1}{2} x^2 \quad \Rightarrow \quad \sin x \approx x$$

In a compact form, we can write the Taylor series for $\cos x$ and $\sin x$ as:

$$\cos x = \sum_{K=0}^{\infty} \frac{x^{2K}}{(2K)!} (-1)^K = \frac{(-1)^0 x^0}{0!} + \frac{(-1)^1 x^2}{2!} + \cdots \approx 1 - \frac{1}{2} x^2 + \cdots$$

$$\sin x = \sum_{K=0}^{\infty} \frac{x^{2K+1}}{(2K+1)!} (-1)^K = \frac{(-1)^0 x^1}{1!} + \frac{(-1)^1 x^3}{3!} + \cdots = x - \frac{x^3}{3!} + \cdots$$
Ex: Taylor Series With Another Base Point

A Taylor series with a base point at $x = b$ (instead of $x = 0$) looks like

$$f(x) = f(b) + f'(b) (x-b) + \frac{f''(b)}{2!} (x-b)^2 + \frac{f'''(b)}{3!} (x-b)^3 + \ldots$$

Ex $f(x) = \sqrt{x}$, $b = 1$. (It is a bad idea to use $b = 0$ because $f(x)$ is not differentiable at $0$).

$$x^{\frac{1}{2}} = 1 + \frac{1}{2} (x-1) + \frac{\frac{1}{2} \left( \frac{1}{2} - 1 \right)}{2!} (x-1)^2 + \ldots$$
Ex: Binomial Expansion: \( f(x) = (1 + x)^a \)

\[
(1 + x)^a = 1 + \frac{a}{1} x + \frac{a(a-1)}{2!} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + \ldots
\]