Polar Coordinates, Area in Polar Coordinates

Polar Coordinates

In polar coordinates, we specify an object's position in terms of its distance \( r \) from the origin and the angle \( \theta \) that the ray from the origin to the point makes with respect to the \( x \) axis.

**Ex:** What are the polar coordinates for the point with rectangular coordinates \((1, -1)\)?

\[ r = \sqrt{1^2 + (-1)^2} = \sqrt{2} \]

\[ \theta = -\frac{\pi}{4} \]

The most common convention is \( r \geq 0 \) and \( 0 \leq \theta < 2\pi \).

Another common convention is \( r \geq 0 \) and \( -\pi \leq \theta < \pi \).

Some conventions use additional restrictions.
No matter what the convention, the following formulas are always true:

- \( X = r \cos \theta \)
- \( Y = r \sin \theta \)

Ex: \((1, -1)\) can be represented by \( r = -\sqrt{2}, \theta = \frac{3\pi}{4} \):

\[ 1 = x = -\sqrt{2} \cos \left( \frac{3\pi}{4} \right), \quad -1 = y = -\sqrt{2} \sin \left( \frac{3\pi}{4} \right) \]

Ex: Consider a circle of radius \( a \) with its center at \((0, 0)\). Let's find an equation that relates \( r \) to \( \theta \).

\[ \text{Circle of radius } a \text{ with center } x = a, y = 0 \]

In rectangular coordinates, the equation for the circle is

\[ (x - a)^2 + y^2 = a^2 \]

We plug in \( X = r \cos \theta, Y = r \sin \theta \):

\[ (r \cos \theta - a)^2 + (r \sin \theta)^2 = a^2 \]

\[ r^2 \cos^2 \theta - 2ar \cos \theta + a^2 + r^2 \sin^2 \theta = a^2 \]

\[ r^2 - 2ar \cos \theta = 0 \]

\[ r = 2a \cos \theta \]
The range of $0 \leq \theta \leq \frac{\pi}{2}$ traces out the top half of the circle, while $-\frac{\pi}{2} \leq \theta \leq 0$ traces out the bottom half. Let's graph this.

Graph of $r = 2a \cos \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

- At $\theta = 0$, $r = 2a \Rightarrow x = 2a$, $y = 0$
- At $\theta = \frac{\pi}{4}$, $r = 2a \cos \frac{\pi}{4} = a\sqrt{2}$

The main issue is finding a range of $\theta$ values that traces the circle once. In this example $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ works.

- $\theta = -\frac{\pi}{2}$ (down)
- $\theta = \frac{\pi}{2}$ (up)
- Area in Polar Coordinates

\[ r = f(\theta) \]

- Let's find the area of a small slice

\[ \text{The small slice is approximately a right triangle.} \]

\[ \text{Area of slice} \approx \text{Area of right triangle} \]

\[ = \frac{1}{2} (\text{base})(\text{height}) = \frac{1}{2} r (r d\theta) \]

- Total Area = \[ \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta \]
Ex: \( r = 2a \cos \theta \), \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\) (the circle from a previous example)

\[
\text{Area} = \int_{\theta = -\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} (2a \cos \theta)^2 \, d\theta = 2a^2 \int_{\theta = -\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta
\]

used trig id: \(\cos^2 \theta = \frac{1}{2}(1 + \cos(2\theta))\)

\[
= a^2 \int_{\theta = -\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos(2\theta)) \, d\theta = a^2 \int_{\theta = -\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta + a^2 \int_{\theta = -\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2\theta) \, d\theta
\]

\[
= a^2 \left[ \theta + \frac{\sin(2\theta)}{2} \right]_{\theta = -\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi a^2.
\]

Ex: Circle centered at the Origin (radius = a)

\[ r = a \]

\[ \begin{align*}
\bullet & \quad x = r \cos \theta \quad y = r \sin \theta \\
\bullet & \quad x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2
\end{align*} \]

The equation of the circle is \( x^2 + y^2 = a^2 \), so \( r = a \) \( \Rightarrow \)

\[ x = a \cos \theta \quad y = a \sin \theta \]

\[
\text{Area} = \int_{\theta = 0}^{\frac{2\pi}{2}} \frac{1}{2} a^2 \, d\theta = \frac{1}{2} \cdot a^2 \cdot 2\pi = \pi a^2
\]
Ex: A ray.

\[ \theta = b \]

In this case, \( \theta = b \), and the range of \( r \) is \( 0 \leq r < \infty \).

\[ x = r \cos b \quad y = r \sin b \]

Ex: The line \( y = 1 \)

To find the polar coordinate equation, plug in \( y = r \sin \theta \), \( x = r \cos \theta \) and solve for \( r \):

\[ y = 1 \]

\[ r \sin \theta = 1 \]

\[ \Rightarrow r = \frac{1}{\sin \theta} \text{ with } 0 < \theta < \pi \]
Ex. Finding the \((x,y)\) coordinates from \(r = f(\theta)\)

As an example, let's consider \(r = \frac{1}{1 + \frac{1}{2} \sin \theta}\)

- \(r + \frac{1}{2} \sin \theta = 1\)

- Plug in \(r = \sqrt{x^2 + y^2}\), \(\sin \theta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}\)

\[\Rightarrow \sqrt{x^2 + y^2} + \frac{y}{2} = 1\]

\[\Rightarrow \sqrt{x^2 + y^2} = 1 - \frac{y}{2} \Rightarrow x^2 + y^2 = (1 - \frac{y}{2})^2 = 1 - y + \frac{y^2}{4}\]

Finally, \(x^2 + \frac{3y^2}{4} + y = 1\).

This is the equation of an ellipse with one focus at the origin.
Ex. A rose $r = \cos(2\theta)$

The graph looks a bit like a flower.

For the first "petal," $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.