**Parametric Equations**

**Ex:** Alternative way to describe a circle of radius $a$:
- $x = a \cos t$
- $y = a \sin t$

- $a$ is a constant and $t$ is a variable

- There is a relationship between $x$ and $y$:
  \[ x^2 + y^2 = a^2 \cos^2 t + a^2 \sin^2 t = a^2 \]

  \[
  \begin{align*}
  \text{at } t=0, & \quad x = a \cos 0 = a, \quad y = a \sin 0 = 0 \\
  \text{at } t=\frac{\pi}{2}, & \quad x = a \cos \frac{\pi}{2} = 0, \quad y = a \sin \frac{\pi}{2} = a \\
  & \text{For } 0 \leq t \leq \frac{\pi}{2}, \text{ a quarter circle is traced counter-clockwise}
  \end{align*}
  \]

**Ex:** Arc length for the previous example:
- $dx = -a \sin t \, dt$
- $dy = a \cos t \, dt$

- $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(-a \sin t \, dt)^2 + (a \cos t \, dt)^2}$

- $ds = \sqrt{(a \sin t)^2 + (a \cos t)^2} \, dt = a \, dt$
Ex: An ellipse in parametric form

- \( X = 2 \sin t \)
- \( Y = \cos t \)
- \( \frac{X^2}{4} + Y^2 = \sin^2 t + \cos^2 t = 1 \)

The ellipse is traced out clockwise as \( t \) increases from 0 to \( \pi/2 \).
Example: We will compute the arc length of a curve in parametric form.

- \( x = t^2 \) for \( 0 \leq t \leq 1 \)
- \( y = t^3 \)
- \( x^3 = (t^2)^3 = t^6 = (t^3)^2 = y^2 \Rightarrow y = x^{\frac{2}{3}} \), \( 0 \leq x \leq 1 \)

- \( dx = 2t \, dt \)
- \( dy = 3t^2 \, dt \)

\[
\begin{align*}
\text{d}s^2 &= dx^2 + dy^2 \\
&= (2t \, dt)^2 + (3t^2 \, dt)^2 = (4t^2 + 9t^4) \, dt^2
\end{align*}
\]

\[
\text{Length} = \int_{0}^{1} \sqrt{4t^2 + 9t^4} \, dt
\]

\[
= \left[ \frac{1}{27} (4^{\frac{3}{2}} - 9t^2) \right]_{0}^{1}
\]

\[
= \frac{1}{27} \left( 13^{\frac{3}{2}} - 4^{\frac{3}{2}} \right)
\]
Surface Area (Surfaces of revolution)

Suppose the strip of width $ds$ is revolved around the $X$ axis.

The surface area of the thin strip is

$$2\pi y \frac{ds}{\text{circumference}}$$

Side view:
Ex: Revolve \[ x = t^2, \quad y = t^3, \quad 0 \leq t \leq 1 \]
around the x-axis

Curved surface of a trumpet

- \[ ds = t \sqrt{4 + 9t^2} \ dt \]

- \[ \text{Area} = \int 2\pi y \, ds = \int 2\pi \cdot t^3 \cdot t \sqrt{4 + 9t^2} \ dt \]

\[ = 2\pi \int t^4 \sqrt{4 + 9t^2} \ dt \]

To evaluate the integral, use the trig substitution

- \[ t = \frac{2}{3} \tan u \quad dt = \frac{2}{3} \sec^2 u \, du \quad \tan^2 u + 1 = \sec^2 u \]

- \[ \int t^4 (4 + 9t^2)^{1/2} \, dt = \int \left( \frac{2}{3} \tan u \right)^4 \left[ 4 + 9 \left( \frac{2}{3} \tan u \right)^2 \right]^{1/2} \frac{2}{3} \sec^2 u \, du \]

\[ = \left( \frac{2}{3} \right)^5 \int \tan^4 u \ (2 \sec u) \ sec^2 u \, du \]

- This is a \( \tan \cdot \sec \) integral. In principle, you could compute the integral, but it would take a long time.
Ex: Let's compute the surface area of a portion of the unit sphere.

1. \( y = \sqrt{1 - x^2} \)

2. \( ds = \frac{dx}{\sqrt{1-x^2}} \) (previously calculated)

3. \( \text{Area} = \int_{a}^{b} 2\pi y \, ds = \int_{a}^{b} 2\pi \sqrt{1-x^2} \frac{dx}{\sqrt{1-x^2}} = \int_{a}^{b} 2\pi \, dx = 2\pi (b-a) \).

For the whole sphere: \( a = -1, b = 1 \)

\[ \text{Area} = 2\pi (1 - (-1)) = 4\pi \]