Partial Fractions

Definition of a rational function: \( \frac{P(x)}{Q(x)} \)

- \( P(x), Q(x) \) both polynomials

**Goal:** Compute \( \int \frac{P(x)}{Q(x)} \, dx \)

\[ \text{Ex: } \frac{1}{x-1} + \frac{3}{x+2} = \frac{(x+2) + 3(x-1)}{(x-1)(x+2)} = \frac{4x-1}{x^2 + x - 2} \]

Therefore:

\[ \int \frac{4x-1}{x^2 + x - 2} \, dx = \int \left( \frac{1}{x-1} + \frac{3}{x+2} \right) \, dx \]

\[ = \ln |x-1| + 3\ln |x+2| + C \]

**Big idea:** in general, split \( \frac{P(x)}{Q(x)} \) into simpler pieces
Ex: How to split \( \frac{4x-1}{x^2+x-2} \) into simpler pieces:

First, factor the denominator: \( x^2+x-2 = (x-1)(x+2) \)

Then guess: \( \frac{4x-1}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2} \)

and find \( A, B \)

Slow way of finding \( A, B \): Clear all denominators by multiplying by \( (x-1)(x+2) \):

\[
4x-1 = A(x+2) + B(x-1)
\]

Then set the coefficients of the various powers of \( x \) on each side equal to each other:

\[
4 = A + B \\
-1 = 2A - B
\]

Then solve for \( A, B \)
Faster way of solving for \( A, B \): “Cover-up” method:

First multiply both sides by \((x-1)\):

\[
\frac{4x-1}{(x-1)(x+2)} = A + \frac{B(x-1)}{x+2}
\]

Then set \( x=1 \) to make the \( B \) term on the right-hand side drop out:

\[
\frac{4-1}{1+2} = A + B \cdot 0 \implies A = 1
\]

Then multiply both sides by \( x+2 \) and set \( x = -2 \) to make the \( A \) term drop out:

\[
\frac{4x-1}{(x-1)(x+2)} = A + \frac{B(x+2)}{x-1}
\]

\[
\frac{4(-2)-1}{-2-1} = B \implies B = 3
\]

This method works when \( Q(x) \) factors into distinct factors and the degree of \( P \) is less than the degree of \( Q \).
If the factors of 0 repeat, we slightly modify the approach:

\[
\frac{x^2 + 2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}
\]

Use the cover up method on the highest degree term in \(x-1\): multiply both sides by \((x-1)^2\)

\[
\frac{x^2 + 2}{x+2} = B + \text{stuff} \cdot (x-1)
\]

Set \(x=1\): \(\frac{1^2 + 2}{1+2} = B \Rightarrow B = 1\)

\(C\) can also be evaluated by the cover up method: multiply both sides by \(x+2\) and set \(x = -2\):

\[
\frac{x^2 + 2}{(x-1)^2} = C + \text{stuff} \cdot (x+2)
\]

\[
\frac{(-2)^2 + 2}{(-3)^2} = C \Rightarrow C = \frac{2}{3}
\]

So far we have:

\[
\frac{x^2 + 2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{1}{(x-1)^2} + \frac{2}{3(x+2)}
\]

Cover up cannot be used to evaluate \(A\). Instead, plug in a convenient \(x\) value into both sides; \(x = 0\)

\[
\frac{2}{(-1)^2 \cdot 2} = \frac{A}{-1} + \frac{1}{(-1)^2} + \frac{2}{3 \cdot 2} \Rightarrow A = \frac{1}{3}
\]

In total:

\[
\frac{x^2 + 2}{(x-1)^2(x+2)} = \frac{1}{3(x-1)} + \frac{1}{(x-1)^2} + \frac{2}{3(x+2)}
\]
$$\Rightarrow \int \frac{x^2+2}{(x-1)^2(x+2)} \, dx = \frac{1}{3} \ln|x+1| - \frac{1}{(x-1)} + \frac{2}{3} \ln|x+2| + C$$

- Not all polynomials factor completely without resorting to complex numbers.

- Example:
  $$\frac{1}{(x^2+1)(x-1)} = \frac{A_1}{x-1} + \frac{B_1 x + C_1}{x^2+1}$$
  
  We find $A_1$ by using the Cover up method (multiply both sides by $x-1$ and set $x=1$):
  $$\frac{1}{1^2+1} = A_1 \Rightarrow \boxed{A_1 = \frac{1}{2}}$$

  - We now have
    $$\frac{1}{(x^2+1)(x-1)} = \frac{\frac{1}{2}}{x-1} + \frac{B_1 x + C_1}{x^2+1}$$

  - To find $C_1$, plug in $x=0$:
    $$\frac{1}{C_1(-1)} = \frac{\frac{1}{2}}{-1} + \frac{C_1}{1} \Rightarrow \boxed{C_1 = -\frac{1}{2}}$$

  - To find $B_1$, plug in any $x$ value except 0 or 1:
    $$x = -1: \quad \frac{1}{2} \frac{-1}{-2} + \frac{B_1(-1) - \frac{1}{2}}{-2} \Rightarrow \boxed{B_1 = -\frac{1}{2}}$$
\[
\int \frac{dx}{(x^2 + 1)(x - 1)} = \frac{1}{2} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{x \, dx}{x^2 + 1} - \frac{1}{2} \int \frac{dx}{x^2 + 1} \\
= \frac{1}{2} \ln |x-1| - \frac{1}{4} \ln |x^2 + 1| - \frac{1}{2} \tan^{-1} x + C
\]

- How to integrate \( \int \frac{dx}{(x+1)^{10}} \)?

\[
\int \frac{dx}{(x-1)^9} = -\frac{1}{9} (x-1)^{-9} + C
\]

- How to integrate \( \int \frac{dx}{(x^2 + 1)^{10}} \)?

  - Use inverse trig substitution
    - \( x = \tan u \) \quad \text{dx} = \sec^2 u \, du \quad \tan^2 u + 1 = \sec^2 u

\[
\int \frac{dx}{(x^2 + 1)^{10}} = \int \frac{\sec^2 u \, du}{(\sec^2 u)^{10}} = \int \cos^{18} u \, du
\]

Could be evaluated using previously discussed methods.