Applications of FTC2 to logarithms

We will now use FTC2 to provide an alternate approach to studying the function \( \ln(x) \).

We introduce the "new" function

\[
L(x) = \int_{1}^{x} \frac{dt}{t}
\]

Note: \( L(1) = 0 \)

Then FTC2 implies

\[
L'(x) = \frac{1}{x}
\]

Recall that \( \frac{d}{dx} \ln x = \frac{1}{x} \) and \( \ln(1) = 0 \).

Thus, by the Mean Value Theorem argument:

\[
\frac{d}{dx} \left( L(x) - \ln(x) \right) = 0, \quad \text{and hence}
\]

\[
L(x) = \ln(x) + C. \quad \text{The constant} \ C \ \text{must be} \ 0 \quad \text{since} \ L(1) = \ln(1) = 0.
\]

Hence,

\[
L(x) = \ln(x)
\]
We can derive some important properties of \( \ln(x) \) by using the representation

\[
L(x) = \int_{1}^{x} \frac{dt}{t}
\]

Claim 1: \( L(ab) = L(a) + L(b) \).

Proof: By the definition of \( L(ab) \) and \( L(a) \), we have

\[
L(ab) = \int_{1}^{ab} \frac{dt}{t} = \int_{1}^{a} \frac{dt}{t} + \int_{a}^{ab} \frac{dt}{t} = L(a) + \int_{a}^{ab} \frac{dt}{t}
\]

We now make the substitution \( t = au \).

Then \( dt = a \, du \) and \( a < t < ab \Rightarrow 1 < u < b \).

Thus,

\[
\int_{a}^{ab} \frac{dt}{t} = \int_{1}^{b} \frac{adu}{au} = \int_{1}^{b} \frac{du}{u} = L(b).
\]

In total: \( L(ab) = L(a) + L(b) \) as desired.
Claim 2: \( L(x) \to \infty \) as \( x \to \infty \).

Proof: We will show that \( L(2^n) \to \infty \) as the integer \( n \to \infty \). Then, since \( L'(x) = \frac{1}{x} \geq 0 \) (when \( x \geq 0 \)), \( L \) is increasing. This fills in the gaps in between the powers of 2.

We use claim 1 to compute:

\[
L(2^n) = \underbrace{L(2 \cdot 2 \cdots 2)}_{n \text{ times}} = \underbrace{L(2) + L(2) + \cdots + L(2)}_{n \text{ times}} = nL(2).
\]

Hence, since \( nL(2) \to \infty \) as \( n \to \infty \), so does \( L(2^n) \).

Claim 3: \( L(x) \to -\infty \) as \( x \to 0^+ \).

Proof: \( 0 = L(1) = L(x \cdot \frac{1}{x}) = L(x) + L\left(\frac{1}{x}\right) \) (by claim 1).

Now as \( x \to 0^+ \), \( \frac{1}{x} \to \infty \), and hence Claim 2 implies that \( L\left(\frac{1}{x}\right) \to \infty \). Thus, \( L(x) = -L\left(\frac{1}{x}\right) \to -\infty \) as \( x \to 0^+ \).
We also compute: \( L''(x) = \frac{d}{dx} \left( \frac{1}{x} \right) = -\frac{1}{x^2} \).

Thus, the graph of \( L(x) \) is concave down when \( x > 0 \).

In total, we have shown that the graph of \( L(x) \) is as follows:

We can define \( \ln(x) = L(x) \), define \( e \)

to be the number such that \( L(e) = 1 \),

define \( e^x \) to be the inverse of \( L(x) \),

and define \( a^x = e^{x \cdot L(x)} \).
Applications of FTC1 to Geometry (Volumes + Areas)

- Area between two curves

The area \( A \) between the curves is

\[
A = \int_{a}^{b} (f(x) - g(x)) \, dx
\]

- \( a \) and \( b \) are the "crossing points"
Ex: Find the area in between the region

\[ x = y^2 \quad \text{and} \quad y = x - 2 \]

To find the crossing points, we solve the equation:

\[ y + 2 = x = y^2 \quad \text{for} \quad y. \]

\[ y^2 - y - 2 = 0 \]

\[ (y - 2)(y + 1) = 0 \quad \Rightarrow \quad y = -1, 2. \]

We then solve for the \( x \)-values corresponding to each \( y \) value. These are \( x = 1, 4 \) respectively.

In total, the two crossing points are \( (1, -1) \) and \( (4, 2) \).
Here are two ways to find the area between the curves.

**Hard way: Vertical Slices.** If we use vertical slices, we need to consider two different regions.

1. The area of region I is \( \int_{x=0}^{x=4} \sqrt[3]{x} - (-\sqrt[3]{x}) \, dx = 2 \int_{x=0}^{x=4} \sqrt[3]{x} \, dx = \left[ \frac{4}{3} x^{\frac{3}{2}} \right]_0^4 = \frac{16}{3} \).

2. The area of region II is \( \int_{x=1}^{x=4} \sqrt[3]{x} - (x-2) \, dx = \left[ \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{2} x^2 + 2x \right]_1^4 = \frac{2}{3} \cdot 4^{\frac{3}{2}} - \frac{1}{2} \cdot 4^2 + 2 \cdot 4 - \left( \frac{2}{3} \cdot \frac{1}{2} + 2 \right) = \frac{19}{6} \).

Then

\[ A = \frac{16}{3} + \frac{19}{6} = \frac{9}{2} \]
Easy Way: Horizontal Slices. Using this method, we subtract the left curve from the right one:

\[ A = \int y_{\text{right}} - y_{\text{left}} \, dy = \int_{y=-1}^{y=2} (y^2) - y^2 \, dy \]

\[ = \left[ \frac{y^2}{2} + \frac{1}{3}y^3 \right]_{-1}^{2} = \frac{4}{2} + \frac{8}{3} - \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{9}{2}. \]
Volumes of solids of revolution

Consider the solid of revolution formed by rotating the curve $y = f(x)$ about the $x$-axis (coming out of the page).

We want to figure out the volume of the solid by first figuring out the volume of a slice of width $dx$ and then adding up the volumes of the slices (i.e., integrating $dx$).
Each slice is approximately a disk of width $dx$, radius $y = f(x)$, and cross sectional area $\pi y^2$.

The volume of one slice is hence for $dV = \pi y^2 \, dx = \pi [f(x)]^2 \, dx$

(for a solid of revolution around the $x$-axis).

- We can then integrate $dx$ to find the total volume.

**Ex:** Find the volume of a sphere of radius $a$.

The equation for the upper half circle is $y = f(x) = \sqrt{a^2 - x^2}$.

- If we rotate this half circle about the $x$-axis, we get a sphere of radius $a$.

In total: $V = \int_{-a}^{a} \pi y^2 \, dx = \int_{-a}^{a} \pi (a^2 - x^2) \, dx = \pi a^2 \left[ x - \frac{1}{3} x^3 \right]_{-a}^{a} = \frac{4}{3} \pi a^3$.

By symmetry, we could have integrated from $0$ to $a$ and then doubled the result $V = 2 \int_{0}^{a} \pi (a^2 - x^2) \, dx = 2 \pi \left[ a^2 x - \frac{1}{3} x^3 \right]_{0}^{a} = \frac{4}{3} \pi a^3$. This saves some work.