Differential Equations

Separation of Variables

Goal: Given an equation of the form
\[(*) \quad \frac{dy}{dx} = F(x,y)\]
solve for \(y\) as a function of \(x\). That is, find \(y = f(x)\) so that eqn. \((*)\) holds.

Ex: \(\frac{dy}{dx} = g(x)\). Then \(y = \int g(x) \, dx\).

We consider these types of equations as solved.
Example: \[(\frac{d}{dx} + x) y = 0\] (equivalently: \[\frac{dy}{dx} + xy = 0\]).

- Solving for \(\frac{dy}{dx}\) gives \[\frac{dy}{dx} = -xy\].

The key step is called **separation of variables**.

\[\frac{dy}{y} = -x\,dx\]

All \(y\) dependence on left, all \(x\) dependence on right.

- Now take antiderivatives of both sides:
  \[\int \frac{dy}{y} = -\int x\,dx\]
  \[\ln |y| = -\frac{x^2}{2} + C\]
  \[|y| = e^{C}e^{-\frac{x^2}{2}}\]
  \[y = ae^{-\frac{x^2}{2}}\] (\(a = \pm e^C\))

Remark: Even though \(e^C \neq 0\), all possible choices of \(a\) (including 0) lead to a solution.
In general:
If \( \frac{dy}{dx} = g(x) h(y) \), then

\[
\frac{dy}{h(y)} = g(x) \, dx
\]

Integrate both sides

\[ H(y) = G(x) + C \quad \text{(implicit formula for } y) \]

\[ H(y) = \int \frac{dy}{h(y)} \quad G(x) = \int g(x) \, dx \]

\[ y = H^{-1}(G(x) + C), \text{ where } H^{-1} \]
is the inverse function of \( H \)

In the previous example,

\[ g(x) = -x \quad h(y) = y \]

\[ G(x) = -\frac{x^2}{2} \quad H(y) = \int \frac{dy}{y} = \ln|y| \]
The solution can be thought of as depending on its initial condition.

For example, if \( y(0) = 1 \), then \( y = e^{-\frac{x^2}{2}} \).

If \( y(0) = a \), then \( y = a e^{-\frac{x^2}{2}} \).

Graph of \( y = e^{-\frac{x^2}{2}} \):
Find a graph such that the slope of the tangent line is twice the slope of the ray from \((0,0)\) to \((x,y)\):

- Translate the problem into a differential equation: \(\frac{dy}{dx} = 2 \left( \frac{y}{x} \right)\).
- Now solve: \(\frac{dy}{y} = \frac{2dx}{x}\) (separation of variables).
- \(\ln|y| = 2\ln|x| + C\) (antiderivatives).
- \(|y| = e^C x^2\) (exponentiate and recall that \(e^{2\ln|x|} = x^2\)).
- \(y = \alpha x^2\).
Examples:

- \( y = x^2 \) \((a = 1)\)
- \( y = 2x^2 \) \((a = 2)\)
- \( y = -x^2 \) \((a = -1)\)
- \( y = 0 \times x^2 = 0 \) \((a = 0)\)
- \( y = -2x^2 \) \((a = -2)\)
- \( y = 100x^2 \) \((a = 100)\)
Ex: Find the curves that are perpendicular to the parabolas from the previous example.

Recall, if two lines respectively have slopes $m_1$ and $m_2$, then they are perpendicular if and only if

$$m_2 = -\frac{1}{m_1}$$

Thus,

$$\frac{dy}{dx} = \frac{1}{\text{slope of parabola}} = -\frac{x}{y}$$

Solve:

$$y \, dy = -\frac{x}{2} \, dx \, (\text{separation of variables})$$

$$\frac{y^2}{2} = -\frac{x^2}{4} + c \, (\text{antiderivatives})$$

$$\frac{x^2}{4} + \frac{x^2}{2} = c \, a \, \text{family of ellipses}$$

$$y = \pm \sqrt{\frac{2}{c-x^2}}$$

Ratio of $x$-semi-major axis to $y$-semi-minor axis is $\sqrt{2}$

One of the ellipses