Top 3 Math Classes for Scientists (reflects personal bias):

1. Calculus
2. Linear Algebra
3. Diff Eq

Why is Calculus important?

1. Many "fundamental" "laws" of nature are expressed in terms of the rate of change of one variable with respect to another. And rate of change\[ \rightarrow \text{ derivatives}\]

2. Many "empirical models" are also expressed in terms of rates of change\[ \rightarrow \text{ a model based on experimental data that does not claim to be a fundamental law of nature.}\]

Examples to come...
Today: Derivatives

I) Geometric interpretation

II) Mathematical definition - limiting procedure

III) How to compute (in principle) the derivative of any function $f(x)$

IV) Physical interpretation

I) Geometric interpretation

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**Geometric Definition** of the derivative of $f$ at $x_0$: the slope of the tangent line through $P = (x_0, f(x_0))$. 

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**Secant line** through $P = (x_0, f(x_0))$ and $Q = (x_0 + \Delta x, f(x_0 + \Delta x))$. 

**Tangent line** through $P$ as $\Delta x \to 0$.
The slope of the tangent line at $P$ is the limit of the slopes of the secant lines $PQ$ as $Q \to P$ ($P$ stays fixed).

\[ \Delta f = f(x_0 + \Delta x) - f(x_0) \]

$\Delta f$ is negative in this picture.

\[ \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \]

"difference quotient" (analytic) of the derivative of $f$ at $x_0$.

this is 2: the mathematical definition of the derivative
Consider \( f(x) = \sqrt{x} \) for \( x \geq 0 \).

Given \( x_0 > 0 \), find the equation of the tangent line at the point \( (x_0, f(x_0)) \).

\[
\begin{align*}
\text{Tangent line: } y - y_0 &= m(x - x_0) \\
(x_0, f(x_0)) &= (x_0, y_0)
\end{align*}
\]

High school algebra: the line through \( (x_0, y_0) \) of slope \( m \) can be expressed as

\[
y_0 = \sqrt{x_0} \quad y - y_0 = m (x - x_0)
\]

\( m \) = slope of tangent line at \( x_0 = f'(x_0) \)

\( \Rightarrow \) we need to compute \( f'(x_0) \).

To compute, we use the analytic definition:

\[
f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x_0 + \Delta x} - \sqrt{x_0}}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{\left(\frac{\sqrt{x_0 + \Delta x} + \sqrt{x_0}}{\Delta x}\right)\left(\frac{\sqrt{x_0 + \Delta x} - \sqrt{x_0}}{\Delta x}\right)}{1}
\]

\[
= \lim_{\Delta x \to 0} \frac{1}{2} \frac{\Delta x}{\sqrt{x_0 + \Delta x} + \sqrt{x_0}}
\]

Simplified version:

\[
\lim_{\Delta x \to 0} \frac{\Delta x}{\sqrt{x_0 + \Delta x} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}
\]
Thus, \( m = f'(x_0) = \frac{1}{2\sqrt{x_0}} \).

Conclusion: the tangent line is (whenever \( x_0 > 0 \))

\[
Y - \sqrt{x_0} = \frac{1}{2\sqrt{x_0}} (X - x_0)
\]

Let's show: the \( y \)-intercept of this line is positive when \( x_0 > 0 \).

Recall: the \( y \)-intercept is the "\( y \)-value" when \( x = 0 \):

\[
y_{\text{int}} = \sqrt{x_0} + \frac{1}{2\sqrt{x_0}} (0 - x_0) = \sqrt{x_0} - \frac{1}{2} \sqrt{x_0} = \frac{1}{2} \sqrt{x_0} > 0
\]
(Lots of) Notation

- Assume that \( y = f(x) \) is a function
- Change in \( y = \Delta y = \Delta f = f(x) - f(x_0) \)
  \[ = f(x_0 + \Delta x) - f(x_0) \]
- Difference quotient \( = \frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x} \)

- Derivative (limit as \( \Delta x \to 0 \)):
  \[ \frac{\Delta y}{\Delta x} \to \frac{dy}{dx} \quad \text{(Leibniz)} \]
  \[ \frac{\Delta f}{\Delta x} \to f'(x_0) \quad \text{(Newton)} \]

- Other derivative notation: \( \frac{df}{dx}, f', Df, y' \)
Example: \( f(x) = x^n \quad n=1, 2, 3, \ldots \)

- Compute \( \frac{d}{dx} x^n \)

- The difference quotient: \( \frac{\Delta y}{\Delta x} = \frac{(x_0 + \Delta x)^n - x_0^n}{\Delta x} \)

  - For simplicity, let's just write "\( x \)" instead of "\( x_0 \)"
  
  This is a reasonable simplification, as long as you remember what you are doing.

  - Then \( \frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x} \)

  - Expand: \( (x + \Delta x)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \Delta x^k \)

  - Above, "\( \mathcal{O}(\Delta x^2) \)" is an abbreviation for all of the terms involving \( (\Delta x)^2, (\Delta x)^3 \), and higher order.

  - The fully detailed expansion formula is called the Binomial Theorem. See your book for the details.

  - Now \( \frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x} = \frac{x^n + n x^{n-1} \Delta x + \mathcal{O}(\Delta x^2) - x^n}{\Delta x} = \frac{n x^{n-1} + \mathcal{O}(\Delta x)}{\Delta x} \)

  - Take the limit: \( \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = n x^{n-1} \).

  Thus, \( \frac{d}{dx} x^n = n x^{n-1} \).
Note the additive and constant multiple properties of differentiation:

- \[ \frac{d}{dx} (f(x) + g(x)) = f'(x) + g'(x) \]
- \[ \frac{d}{dx} (c f(x)) = c f'(x) \text{ when } c \text{ is a constant.} \]

You can now compute the derivatives of all polynomials.

**Ex.** \[ \frac{d}{dx} (x^7 + x) = \frac{d}{dx} x^7 + \frac{d}{dx} x = 7x^6 + 1. \]