Problem 1. pg. 160 problem 85

Problem 2. Show that for any two numbers $a$ and $b$, $|\sin a - \sin b| \leq |a - b|$.

Problem 3. Section 4.5: 8.

Problem 4. Use Newton’s method to estimate the zero of $f(x) = x^3 + 5x - 7$. Start with the base point $x_0 = 1$ and compute $x_1, x_2$.

Problem 5. Graph the function $f(x) = |x|^{5/2} - 3|x|^{3/2} + |x|^{1/2}$. Indicate all zeros, critical points, inflection points, points of discontinuity, regions where $f(x)$ is increasing/decreasing, and regions where $f(x)$ is concave up/down.

Problem 6. pg. 156 problem 50

Problem 7. Compute the following antiderivatives:

a) $\int \sin(x^2)x^2(1 + \ln x) \, dx$

b) $\int \frac{\arctan(3x)}{(1 + 9x^2)(1 + [\arctan(3x)]^2)} \, dx$

Problem 8. Consider the function $f(x) = (1 + x)^\alpha [1 + \ln(1 + \beta x)]$, where $\alpha$ and $\beta$ are constants. Find the constants $\alpha$ and $\beta$ that make the graph of $f(x)$ “as flat as possible” near $x = 0$. The choice $\beta = 0$ is forbidden.
Solutions

Problem 1. pg. 160 problem 85

Solution: If the woman runs the distance $L$ along the $x$–axis, then she must swim the distance $\sqrt{b^2 + (L - a)^2}$. The total time she spends to reach the point $(a,b)$ is

$$T = \frac{L}{r} + \frac{\sqrt{b^2 + (L - a)^2}}{s}.$$  

The range of $L$ values under consideration is $0 \leq L$.

To find the critical points of $T$, we first compute

$$\frac{dT}{dL} = \frac{1}{r} + \frac{[b^2 + (L - a)^2]^{-1/2}(L - a)}{s}.$$  

Setting $\frac{dT}{dL} = 0$, we solve for the critical point $L_{\text{critical}}$ as follows:

$$L_{\text{critical}} = a - \frac{b}{\sqrt{\frac{r^2}{s^2} - 1}}.$$  

As long as the above formula leads to $L_{\text{critical}} > 0$, it is straightforward to verify that $\frac{dT}{dL} > 0$ when $L > L_{\text{critical}}$ and $\frac{dT}{dL} < 0$ when $L < L_{\text{critical}}$. Thus, as long as $L_{\text{critical}} > 0$, $L_{\text{critical}}$ is in fact the minimum value.

Problem 2. Show that for any two numbers $a$ and $b$, $|\sin a - \sin b| \leq |a - b|$.  

Solution: Let $f(x) = \sin x$. By the mean value theorem, there exists a point $c$ in between $a$ and $b$ such that $|\sin a - \sin b| = |f'(c)||b - a| = |\cos c||b - a| \leq |b - a|.

Problem 3. Section 4.5: 8.

Solution: Assume that the boy is standing at the origin in the $x,y$ plane and that the kite is at the location $(x,y)$. Let $D$ denote the length of the string. By the pythagorean theorem, we have

$$D^2 = x^2 + y^2.$$  

Using the chain rule, we differentiate each side of the equation with respect to $t$ to deduce that

$$2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}.$$  

We are told that

$$y = 80,$$
$$D = 100,$$

from which it follows that $x = 60$. We are also told that

$$\frac{dx}{dt} = 20,$$
$$\frac{dy}{dt} = 0.$$
Plugging these numbers into the above equation, we deduce that
\[
\frac{dD}{dt} = \frac{x}{D} \frac{dx}{dt} + \frac{y}{D} \frac{dy}{dt} = 60 \times \frac{20 + 0}{100} = 12.
\]

**Problem 4.** Use Newton’s method to estimate the zero of \( f(x) = x^3 + 5x - 7 \). Start with the base point \( x_0 = 1 \) and compute \( x_1, x_2 \).

**Solution:** Newton’s iterate formula is
\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.
\]
Since \( f'(x) = 3x^2 + 5 \), we have
\[
x_{k+1} = x_k - \frac{x_k^3 + 5x_k - 7}{3x_k^2 + 5}.
\]
We then set \( x_0 = 1 \) and compute
\[
x_1 = 1 - \frac{-1}{8} = \frac{9}{8},
\]
\[
x_2 = \frac{9}{8} - \frac{\frac{9^3}{8^3} + 5 \cdot \frac{9}{8} - 7}{3 \cdot \frac{9^2}{8^2} + 5} = \frac{9}{8} - \frac{9^3 + 45 \times 8^2 - 7 \times 8^3}{3 \times 9^2 \times 8 + 5 \times 8^3} = \frac{9}{8} - \frac{25}{4504}.
\]

**Problem 5.** Graph the function \( f(x) = |x|^{5/2} - 3|x|^{3/2} + |x|^{1/2} \). Indicate all zeros, critical points, inflection points, points of discontinuity, regions where \( f(x) \) is increasing/decreasing, and regions where \( f(x) \) is concave up/down.

**Solution:** The function is even, so we only need to consider \( x \geq 0 \). We first note that \( f(x) \to \infty \) as \( x \to \infty \) and \( f(x) \to 0 \) as \( x \to 0^+ \).

We then compute that for \( x > 0 \), we have
\[
f'(x) = \frac{1}{2} \frac{(5x^2 - 9x + 1)}{\sqrt{x}},
\]
\[
f''(x) = \frac{1}{4} \frac{(15x^2 - 9x - 1)}{x^{3/2}}.
\]
To find the critical points in the region \( x > 0 \), we set \( f'(x) = 0 \) and solve via the quadratic formula:
\[
x_{\text{critical}} = \frac{9 \pm \sqrt{61}}{10}.
\]
Note that both of these numbers are positive. In between 0 and \( x_{\text{critical}^-} \), \( f' > 0 \) and so \( f \) is increasing. In between \( x_{\text{critical}^-} \) and \( x_{\text{critical}^+} \), \( f' < 0 \) and so \( f \) is decreasing. In between \( x_{\text{critical}} \) and \( \infty \), \( f' > 0 \) and so \( f \) is increasing. Also, \( f'(x) \) becomes infinite as \( x \to 0^+ \).

To find the inflection points, we set \( f''(x) = 0 \) and solve via the quadratic formula:

\[
x_{\text{inflection}} = \frac{9 + \sqrt{141}}{30}.
\]

Note that we have discarded the other root since it is not positive. In between 0 and \( x_{\text{inflection}} \), \( f'' < 0 \) and so \( f \) is concave down. In between \( x_{\text{inflection}} \) and \( \infty \), \( f'' > 0 \) and so \( f \) is concave up.

The full graph is given in the figure below.

![Figure 1. Graph of \( f(x) \)](image)

Problem 6. pg. 156 problem 50

Solution: We will use the hint in the book. In particular, since the length of the base and the area are given, this implies that the height \( h = 2\text{area}/(\text{length of base}) \) is fixed. Suppose that the vertex has coordinates \((x, h)\). Without loss of generality, we can assume that \( x \geq 0 \) (otherwise, we just flip the triangle about the \( y \) axis). Assume that the two vertices of the base are at \((-a, 0)\) and \((a, 0)\), where \( a \) is a constant. Then by the the pythagorean theorem, the lengths of the other two
sides are
\[
\ell_1 = \sqrt{(x + a)^2 + h^2}, \\
\ell_2 = \sqrt{(x - a)^2 + h^2}.
\]

We therefore want to minimize the function
\[
f(x) = \ell_1 + \ell_2 = \sqrt{(x + a)^2 + h^2} + \sqrt{(x - a)^2 + h^2}
\]
over the region \(x \geq 0\). Clearly \(f(x) \to \infty\) as \(x \to \infty\), so the minimizer will not "lie at \(x = \infty\)."

To locate the critical points of \(f(x)\), we first compute
\[
f'(x) = \frac{x + a}{\sqrt{(x + a)^2 + h^2}} + \frac{x - a}{\sqrt{(x - a)^2 + h^2}}.
\]
We then set \(f'(x) = 0\) to deduce the equation
\[
\frac{x + a}{\sqrt{(x + a)^2 + h^2}} = -\frac{x - a}{\sqrt{(x - a)^2 + h^2}}.
\]

Squaring the equation to make life easier, we deduce
\[
\frac{(x + a)^2}{(x + a)^2 + h^2} = \frac{(x - a)^2}{(x - a)^2 + h^2},
\]
which is equivalent to
\[
\frac{1}{1 + \frac{h^2}{(x + a)^2}} = \frac{1}{1 + \frac{h^2}{(x - a)^2}}.
\]
We then see that
\[
(x + a)^2 = (x - a)^2.
\]

The above equation has only the solution \(x = 0\). Thus, the only critical point is also an endpoint. Therefore, \(x = 0\) must be the minimum. Since \(x = 0\) implies that the triangle is isosceles, we have proved the desired result.

**Problem 7.** Compute the following antiderivatives:

a) \[\int \sin(x^x)x^x(1 + \ln x) \, dx\]

b) \[\int \frac{\arctan(3x)}{(1 + 9x^2)\sqrt{1 + [\arctan(3x)]^2}} \, dx\]

**Solution:** a) We set \(u = x^x\). This implies (by logarithmic differentiation) that \(du = x^x(1 + \ln x) \, dx\). After these substitutions, the integral becomes
\[
\int \sin u \, du = -\cos u + c = -\cos(x^x) + c.
\]

b) We first make the substitution \(u = \arctan(3x), du = 3(1 + 9x^2)^{-1} \, dx\), which leads to the integral
\[
\frac{1}{3} \int \frac{u}{\sqrt{1 + u^2}} \, du.
\]
We then make the second substitution \( v = u^2 \), \( dv = 2u \, du \), and the integral becomes
\[
\frac{1}{6} \int \frac{dv}{\sqrt{1 + v}} \, dv = \frac{1}{6} \int (1 + v)^{-1/2} (1 + u^2)^{1/2} + c
\]
\[
= \frac{1}{3}(1 + u^2)^{1/2} + c
\]
\[
= \frac{1}{3}(1 + [\text{arctan}(3x)]^2)^{1/2} + c.
\]

**Problem 8.** Consider the function \( f(x) = (1 + x)\alpha [1 + \ln(1 + \beta x)] \), where \( \alpha \) and \( \beta \) are constants.
Find the constants \( \alpha \) and \( \beta \) that make the graph of \( f(x) \) “as flat as possible” near \( x = 0 \). The choice \( (\alpha, \beta) = (0, 0) \) is forbidden.

**Solution:** We first compute the quadratic approximation to \( f(x) \):
\[
f(x) = \left(1 + \alpha x + \frac{\alpha(\alpha - 1)x^2}{2} + O(x^3)\right) \left(1 + \beta x - \frac{\beta^2 x^2}{2} + O(x^3)\right)
\]
\[
= 1 + (\alpha + \beta)x + \left(\alpha\beta + \frac{\alpha(\alpha - 1)}{2} - \frac{\beta^2}{2}\right)x^2 + O(x^3).
\]
To make the graph of \( f(x) \) as flat as possible, we set the coefficients of \( x \) and \( x^2 \) equal to 0:
\[
\alpha + \beta = 0,
\]
\[
\alpha\beta + \frac{\alpha(\alpha - 1)}{2} - \frac{\beta^2}{2} = 0.
\]
The first equation implies that \( \alpha = -\beta \). Inserting this information into the second equation, we deduce
\[
-\alpha^2 - \frac{1}{2}\alpha = 0.
\]
This equation has the forbidden solution \( \alpha = 0 \) (forbidden because it leads to \( \beta = 0 \)) and also the solution \( \alpha = -1/2 \). Thus,
\[
(\alpha, \beta) = (-1/2, 1/2),
\]
and \( f(x) = (1 + x)^{-1/2} \left[1 + \ln(1 + \frac{1}{2}x)\right] \).