LT. Laplace Transform

1. Translation formula. The usual L.T. formula for translation on the \( t \)-axis is

\[
\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s), \quad \text{where} \quad F(s) = \mathcal{L}(f(t)), \quad a > 0.
\]

This formula is useful for computing the inverse Laplace transform of \( e^{-as}F(s) \), for example. On the other hand, as written above it is not immediately applicable to computing the L.T. of functions having the form \( u(t-a)f(t) \). For this you should use instead this form of (1):

\[
\mathcal{L}(u(t-a)f(t)) = e^{-as}\mathcal{L}(f(t+a)), \quad a > 0.
\]

Example 1. Calculate the Laplace transform of \( u(t-1)(t^2+2t) \).

Solution. Here \( f(t) = t^2 + 2t \), so (check this!) \( f(t+1) = t^2 + 4t + 3 \). So by (2),

\[
\mathcal{L}(u(t-1)(t^2+2t)) = e^{-s}L(t^2 + 4t + 3) = e^{-s} \left( \frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s} \right).
\]

Example 2. Find \( \mathcal{L}(u(t-\pi/2) \sin t) \).

Solution. \( \mathcal{L}(u(t-\pi/2) \sin t) = e^{-\pi s/2}L(\sin(t + \pi/2)) = e^{-\pi s/2}L(\cos t) = e^{-\pi s/2} \frac{s}{s^2 + 1}. \)

Proof of formula (2). According to (1), for any \( g(t) \) we have

\[
\mathcal{L}(u(t-a)g(t-a)) = e^{-as}\mathcal{L}(g(t));
\]

this says that to get the factor on the right side involving \( g \), we should replace \( t-a \) by \( t \) in the function \( g(t-a) \) on the left, and then take its Laplace transform.

Apply this procedure to the function \( f(t) \), written in the form \( f(t) = f((t-a)+a) \); we get (“replacing \( t-a \) by \( t \) and then taking the Laplace Transform’’)

\[
\mathcal{L}(u(t-a)f((t-a)+a)) = e^{-as}\mathcal{L}(f(t+a)),
\]

exactly the formula (2) that we wanted to prove. \( \square \)

Exercises. Find: a) \( \mathcal{L}(u(t-a)e^t) \) b) \( \mathcal{L}(u(t-\pi) \cos t) \) c) \( \mathcal{L}(u(t-2)/e^{-t}) \)

Solutions. a) \( e^{-as} \frac{e^a}{s-1} \) b) \( -e^{-\pi s} \frac{s}{s^2 + 1} \) c) \( e^{-2s} \frac{e^{-2}(2s + 3)}{(s+1)^2} \)