18.03 Solutions

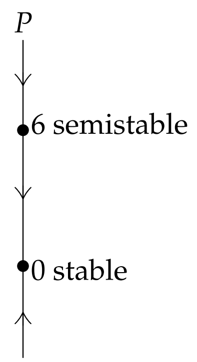
8: Extra Problems

8A. Bifurcation Diagrams

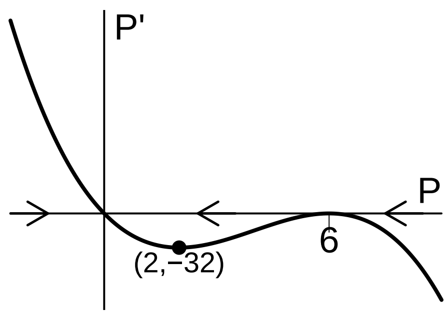
8A-1.

a) Critical points: $f(P) = 0 \Rightarrow P = 0, 6$. See below for phase line. The integral curves are not shown. Make sure you know how to sketch them as functions of P vs. t.

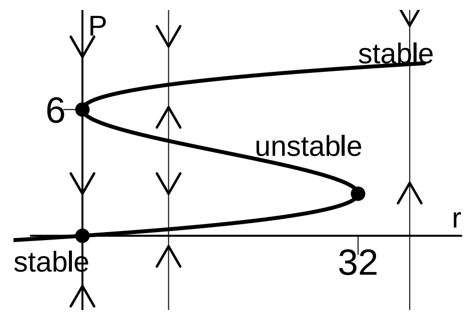
b) The picture below shows the graph of $P' = f(P)$ (i.e. when $r = 0$).
A positive r will raise the graph. As soon as $r > 32$ the graph will have only one zero and that zero will be above 6. **Solution.** $r = 32$.



Phaseline ($r = 0$)



P' vs. P (r=0)



Bifurcation

c) See above diagram. The curve of critical points is given by solving

$$P' = -P^3 + 12P^2 - 36P + r = 0 \Rightarrow r = P^3 - 12P^2 + 36P,$$

which is a sideways cubic. The phase-line for $r = 0$ is determined by the middle plot. The phase line for the other values of r then follow by continuity, i.e. the rP-plane is divided into two pieces by the curve, and arrows in the same piece have to point the same way.

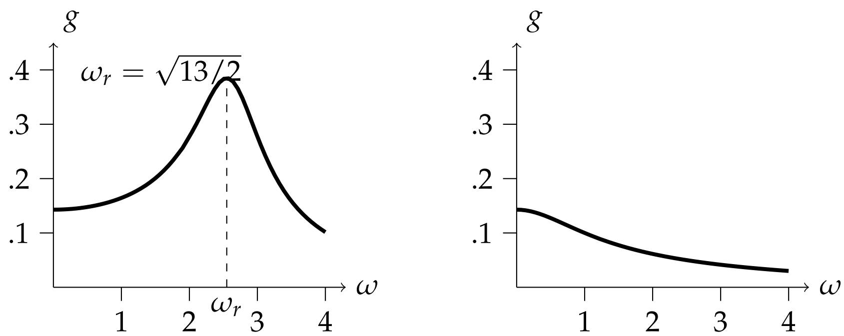
8B. Frequency Response

8B-1.

a) Characteristic polynomial: $p(s) = r^2 + r + 7$
Complexified ODE: $\tilde{x}'' + \tilde{x} + 7\tilde{x} = F_0 e^{i\omega t}$.
Particular solution (from Exp. Input Theorem): $\tilde{x}_p = F_0 e^{i\omega t}/p(i\omega) = F_0 e^{i\omega t}/(7 - \omega^2 + i\omega)$
Complex and real gain: $\tilde{g}(\omega) = 1/(7 - \omega^2 + i\omega)$, $g(\omega) = 1/|p(i\omega)| = 1/\sqrt{(7-\omega^2)^2 + \omega^2}$.
For graphing we analyze the term under the square root: $f(\omega) = (7 - \omega^2)^2 + \omega^2$.
Critical points: $f'(\omega) = -4\omega(7 - \omega^2) + 2\omega = 0 \Rightarrow \omega = 0$ or $\omega = \sqrt{13/2}$.
Evaluate at the critical points: $g(0) = 1/7$, $g(\sqrt{13/2}) = .385$
Find regions of increase and decrease by checking values of $f'(\omega)$:
On $[0, \sqrt{13/2}]$: $f(\omega) < 0 \Rightarrow f$ is decreasing $\Rightarrow g$ is increasing.
On $[\sqrt{13/2}, \infty]$: $f(\omega) > 0 \Rightarrow f$ is increasing $\Rightarrow g$ is decreasing.

The graph is given below.
This system has a (practical) resonant frequency = $\omega_r = \sqrt{13/2}$.

b) Characteristic polynomial: $p(s) = r^2 + 8r + 7$
Complex and real gain: $\tilde{g}(\omega) = 1/p(i\omega) = 1/(7 - \omega^2 + i8\omega)$, $g(\omega) = 1/|p(i\omega)| = 1/\sqrt{(7-\omega^2)^2 + 64\omega^2}$.
For graphing we analyze the term under the square root: $f(\omega) = (7-\omega^2)^2 + 64\omega^2$.
Critical points: $f'(\omega) = -4\omega(7-\omega^2) + 128\omega = 0 \Rightarrow \omega = 0$.
Since there are no positive critical points the graph is strictly decreasing.
Graph below.



Graphs for 8B-1a and 8B-1b.

8C. Pole Diagrams

8C-1.

a) All poles have negative real part: a, b, c, h.

b) All poles have nonzero imaginary part: b, d, e, f, h.

c) All poles are real: a, g.

d) Poles are real or complex conjugate pairs: a, b, c, g, h.

e) b, because the pole farthest to the right in b, is more negative than the one in c.

f) This is just the number of poles: a) 2, b) 2, c) 4, d) 2, e) 2, f) 4, g) 3, h) 2.

g) a) Making up a scale, the poles are -1 and -3 $\Rightarrow P(s) = (s+1)(s+3) \Rightarrow P(D) = D^2 + 4D + 3$.
b) Possible poles are $-3 \pm 2i \Rightarrow P(s) = (s+3-2i)(s+3+2i) \Rightarrow P(D) = D^2 + 6D + 13$.
c) Possible poles are $-1, -3, -2 \pm 2i \Rightarrow P(s) = (s+1)(s+3)(s+2-2i)(s+2+2i)$
$\Rightarrow P(D) = (D+1)(D+3)(D^2+4D+8) = D^4 + 8D^3 + 27D^2 + 44D + 24$.

h) System (h). The amplitude of the reponse is $1/|P(i\omega)|$. In the pole diagram $i\omega$ is on the imaginary axis. The poles represent values of s where $1/P(s)$ is infinite. The poles in system (h) are closer to the imaginary axis than those in system (b), so the biggest $1/|P(i\omega)|$ is bigger in (h) than (b).

M.I.T. 18.03 Ordinary Differential Equations
18.03 Notes and Exercises