

Section 3 Solutions

3A-1 $\mathcal{L}\{t\} = \int_0^{\infty} t e^{-st} dt$. Integrate by parts:

$$= t \frac{e^{-st}}{-s} \Big|_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} dt$$

Since $\lim_{t \rightarrow \infty} t e^{-st} = 0$ if $s > 0$, the left-hand term is 0 at both endpoints. Integrating the right-hand term:

$$= -\frac{e^{-st}}{(-s)(-s)} \Big|_0^{\infty} = 0 - \left(-\frac{1}{s^2}\right) = \frac{1}{s^2}, \quad s > 0.$$

3A-4 $\mathcal{L}\{\sin at\} = \int_0^{\infty} \sin at \cdot e^{-st} dt$; Integrate by parts:

$$= \sin at \cdot \frac{e^{-st}}{-s} \Big|_0^{\infty} - \int_0^{\infty} a \cos at \cdot \frac{e^{-st}}{-s} dt$$

$$= 0 + \frac{a}{s} \mathcal{L}\{\cos at\}$$

$$= \frac{a}{s} \cdot \frac{s}{s^2 + a^2}, \quad s > 0$$

$$= \frac{a}{s^2 + a^2}, \quad s > 0.$$

3A-2 $\mathcal{L}\{e^{(a+ib)t}\} = \mathcal{L}\{e^{at} \cos bt\} + i \mathcal{L}\{e^{at} \sin bt\}$ (*)

On the other hand,
 $\mathcal{L}\{e^{(a+ib)t}\} = \frac{1}{s - (a+ib)}$; multiplying top + bottom by $(s-a) + ib$:

$$= \frac{(s-a) + ib}{(s-a)^2 + b^2} = \frac{s-a}{(\dots)} + \frac{ib}{(\dots)} \quad (**)$$

$$\therefore \mathcal{L}\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2}, \quad \mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

[by equating real + imag. parts of (*) and (**).]

3A-5 $\mathcal{L}\{\cos^2 at\} = \mathcal{L}\{\frac{1}{2} + \frac{1}{2} \cos 2at\}$

$$= \mathcal{L}\{\frac{1}{2}\} + \frac{1}{2} \mathcal{L}\{\cos 2at\}$$

$$= \frac{1}{2s} + \frac{1}{2} \left(\frac{s}{s^2 + 4a^2} \right)$$

$$\mathcal{L}\{\sin^2 at\} = \mathcal{L}\{\frac{1}{2} - \frac{1}{2} \cos 2at\}$$

$$= \frac{1}{2s} - \frac{1}{2} \left(\frac{s}{s^2 + 4a^2} \right)$$

$$\mathcal{L}\{\cos^2 at + \sin^2 at\} = \frac{1}{s}, \quad \text{from the above;}$$

$$\mathcal{L}\{1\} = \frac{1}{s} \quad \checkmark$$

3A-3 a) $\mathcal{L}^{-1}\left(\frac{1}{\frac{s}{2} + 3}\right) = \mathcal{L}^{-1}\left(\frac{2}{s+6}\right) = 2e^{-6t}$

b) $\mathcal{L}^{-1}\left(\frac{3}{s^2+4}\right) = \frac{3}{2} \mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = \frac{3}{2} \sin 2t$

c) $\mathcal{L}^{-1}: \frac{1}{s^2-4} = \frac{1/4}{s-2} - \frac{1/4}{s+2}$ (partial fractions)

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^2-4}\right) = \frac{1}{4} e^{2t} - \frac{1}{4} e^{-2t}$$

d) $\frac{1+2s}{s^2} = \frac{1}{s^2} + \frac{2}{s}$

$$\therefore \mathcal{L}^{-1}\left(\frac{1+2s}{s^2}\right) = \frac{t^2}{2} + 2t$$

e) $\frac{1}{s^4-9s^2} = \frac{-1/9}{s^2} + \frac{0}{s} + \frac{1/54}{s-3} + \frac{-1/54}{s+3}$

$$= \frac{1}{s^2(s-3)(s+3)}$$

(by cover-up method. Find the 0 by putting $s=1$)

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^4-9s^2}\right) = -\frac{t}{9} + \frac{1}{54} (e^{3t} - e^{-3t})$$

3A-6a $\mathcal{L}\left\{\frac{1}{\sqrt{t}}}\right\} = \int_0^{\infty} e^{-st} \frac{1}{\sqrt{t}} dt, \quad (s > 0)$

Put $x^2 = st$, so $t = \frac{x^2}{s}$

Then the integral becomes (in terms of s, x):

$$= \int_0^{\infty} e^{-x^2} \frac{\sqrt{s}}{x} \cdot \frac{2x dx}{s}$$

$$= \frac{2}{\sqrt{s}} \int_0^{\infty} e^{-x^2} dx = \frac{2}{\sqrt{s}} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

b) $\mathcal{L}\{\sqrt{t}\} = \int_0^{\infty} e^{-st} \sqrt{t} dt$; integrate by parts:

$$= \sqrt{t} \frac{e^{-st}}{-s} \Big|_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} \cdot \frac{1}{2\sqrt{t}} dt$$

$$= 0 + \frac{1}{2s} \int_0^{\infty} e^{-st} \cdot \frac{1}{\sqrt{t}} dt$$

$$= \frac{1}{2s} \mathcal{L}\left\{\frac{1}{\sqrt{t}}}\right\} = \frac{1}{2s} \cdot \frac{\sqrt{\pi}}{\sqrt{s}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$\boxed{3A-7} \quad \mathcal{L}\{e^{t^2}\} = \int_0^{\infty} e^{-st} \cdot e^{t^2} dt$$

$$= \int_0^{\infty} e^{t^2-st} dt$$

This integral is infinite for every real value of s , no matter how large, since if $t > s$, $t^2-st > 0$, and therefore

$$\int_0^{\infty} e^{t^2-st} dt > \int_s^{\infty} e^{t^2-st} dt > \int_s^{\infty} e^0 dt, \quad \infty.$$

$$\boxed{3A-8} \quad \mathcal{L}\left\{\frac{1}{t^k}\right\} = \int_0^{\infty} e^{-st} \frac{1}{t^k} dt, \quad (s > 0)$$

The trouble here is when $t=0$. Near $t=0$, $e^{-st} \approx e^0 = 1$.

\therefore the integral is like:

$$\int_0^a e^{-st} \frac{1}{t^k} dt \gtrsim \int_0^a \frac{dt}{t^k}$$

and this last integral converges only if $k < 1$ [since it $\left\{ \begin{array}{l} = \frac{t^{-k}}{-k} \Big|_0^a \text{ for } k \neq 1 \\ = \ln x \Big|_0^a \text{ for } k = 1 \end{array} \right.$]

[At the upper limit ∞ the original integral always converges, if $s > 0$].

$\therefore \mathcal{L}\left\{\frac{1}{t^k}\right\}$ exists for $k < 1$.

$$\boxed{3A-9a} \quad \mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9} = F(s)$$

By the exponential-shift formula,

$$\mathcal{L}\{e^{-t} \sin 3t\} = F(s+1) = \frac{3}{(s+1)^2+9}$$

$$b) \quad \mathcal{L}\{t^2-3t+2\} = \frac{2}{s^3} - \frac{3}{s^2} + \frac{2}{s} = F(s)$$

By exponential-shift rule,

$$\mathcal{L}\{e^{2t}(t^2-3t+2)\} = F(s-2)$$

$$= \frac{2}{(s-2)^3} - \frac{3}{(s-2)^2} + \frac{2}{s-2}$$

$$\boxed{3A-10} \quad \mathcal{L}^{-1}\left\{\frac{3}{(s-2)^4}\right\} = e^{2t} \mathcal{L}^{-1}\left\{\frac{3}{s^4}\right\} = e^{2t} \frac{t^3}{2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s-2} - \frac{1/2}{s}\right\},$$

(by partial fractions)

$$= \frac{1}{2} e^{2t} - \frac{1}{2}$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s+5}\right\} :$$

Complete the square in the denominator:

$$\frac{s+1}{s^2-4s+5} = \frac{s+1}{(s-2)^2+1} ; \quad \begin{array}{l} \text{express} \\ \text{top in} \\ \text{terms of } s-2. \end{array}$$

$$= \frac{s-2}{(s-2)^2+1} + \frac{3}{(s-2)^2+1}$$

$$\therefore \mathcal{L}^{-1}(\dots) = e^{2t} \cos t + 3e^{2t} \sin t,$$

(by the exponential-shift rule).

3B-1

We use throughout the two formulas:

$$\mathcal{L}(y') = -y(0+) + sY \leftarrow \mathcal{L}(y)$$

and

$$\mathcal{L}(y'') = -y'(0+) - sy(0+) + s^2 Y$$

[The 0+ indicates that if $y(t)$ is discontinuous at 0, we $\lim_{t \rightarrow 0+} y(t)$, the righthand limit].

a) $y' - y = e^{3t}, \quad y(0) = 1$

$$(sY - 1) - Y = \frac{1}{s-3}$$

$$(s-1)Y = 1 + \frac{1}{s-3}$$

$$Y = \frac{1}{s-1} + \frac{1}{(s-3)(s-1)}$$

make partial fractions decomp;

$$= \frac{1/2}{s-1} + \frac{1/2}{s-3}$$

$$\therefore y = \frac{1}{2}e^t + \frac{1}{2}e^{3t}$$

b) $y'' - 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$

$$(s^2 Y - s - 1) - 3(sY - 1) + 2Y = 0$$

$$\therefore (s^2 - 3s + 2)Y = s - 2$$

$$Y = \frac{1}{s-1}$$

$$\therefore y = e^t$$

c) $y'' + 4y = \sin t, \quad y(0) = 1, \quad y'(0) = 0$

$$(s^2 Y - s) + 4Y = \frac{1}{s^2 + 1}$$

$$\therefore Y = \frac{1}{(s^2 + 1)(s^2 + 4)} + \frac{s}{s^2 + 4} \quad (*)$$

Apply partial fractions \uparrow ; treat s^2 as a single variable: i.e.,

$$\frac{1}{(u+1)(u+4)} = \frac{1/3}{u+1} - \frac{1/3}{u+4}; \quad \text{now put } u = s^2$$

$$Y = \frac{1/3}{s^2 + 1} - \frac{1/3}{s^2 + 4} + \frac{s}{s^2 + 4}$$

$$\therefore y = \frac{1}{3} \sin t - \frac{1}{6} \sin 2t + \cos 2t$$

(*) Note that it's easier not to combine terms at this point

d) $y'' - 2y' + 2y = 2e^t, \quad y(0) = 0, \quad y'(0) = 1$

$$(s^2 Y - 1) - 2sY + 2Y = \frac{2}{s-1}$$

$$\therefore (s^2 - 2s + 2)Y = \frac{2}{s-1} + 1 = \frac{s+1}{s-1}$$

$$Y = \frac{s+1}{(s^2 - 2s + 2)(s-1)}$$

By partial fractions:

$$Y = \frac{2}{s-1} + \frac{3-2s}{s^2 - 2s + 2}; \quad \text{complete the square:}$$

$$= \frac{2}{s-1} - \frac{2(s-1)}{(s-1)^2 + 1} + \frac{1}{(s-1)^2 + 1}$$

(note how we write the 2nd term as an expression in $s-1$; the last term is what's left over.)

$$\therefore y = 2e^t - 2e^t \cos t + e^t \sin t$$

e) $y'' - 2y' + y = e^t, \quad y(0) = 1, \quad y'(0) = 0$

$$s^2 Y - s - 2(sY - 1) + Y = \frac{1}{s-1}$$

$$(s^2 - 2s + 1)Y = \frac{1}{s-1} + s - 2$$

$$\frac{1}{(s-1)^2} = \frac{1}{s-1} + (s-1) \cdot -1$$

$$\therefore Y = \frac{1}{(s-1)^3} + \frac{1}{s-1} - \frac{1}{(s-1)^2}$$

$$\therefore y = \frac{t^2}{2} e^t + e^t - t e^t$$

3B-2

12. $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt$

Integ. by parts: $= e^{-st} f(t) \Big|_0^\infty - \int_0^\infty -s e^{-st} f(t) dt$

see below: (since $f(t)$ is of exp. order) $\Rightarrow 0 - f(0) + s \int_0^\infty e^{-st} f(t) dt$

$$\therefore \mathcal{L}\{f'(t)\} = -f(0) + s \mathcal{L}\{f(t)\}$$

Assumes:

$f(t)$ piecewise continuous and exponential order (so $\int_0^\infty e^{-st} f(t) dt$ exists) (i.e., $|f(t)| \leq K e^{at}$ if t is large).

$f(t)$ of exponential order, so $\mathcal{L}\{f\}$ exists. (It's continuous, since $f'(t)$ exists).

3B-3

These use the formula:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

↑
= $\mathcal{L}\{f(t)\}$

a) $\mathcal{L}\{t \cos bt\} = (-1) \frac{d}{ds} \left(\frac{s}{s^2+b^2} \right)$
 $= \frac{b^2 - s^2}{(b^2 + s^2)^2}$

b) $\mathcal{L}\{t^n e^{kt}\}$: by the exp-shift rule,
 $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
 $\therefore \mathcal{L}\{t^n e^{kt}\} = \frac{n!}{(s-k)^{n+1}}$

By the above formula,

$$\begin{aligned} \mathcal{L}\{t^n e^{kt}\} &= (-1)^n \frac{d^n}{ds^n} (s-k)^{-1} \\ &= (-1)^n \cdot (-1)(-2) \cdots (-n) (s-k)^{-(n+1)} \\ &= \frac{n!}{(s-k)^{n+1}}, \text{ as before.} \end{aligned}$$

c) $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

$$\mathcal{L}\{t \sin t\} = \frac{2s}{(s^2+1)^2} \text{ by the above formula.}$$

$$\therefore \mathcal{L}\{t e^{at} \sin t\} = \frac{2(s-a)}{(s-a)^2+1)^2}$$

3B-4

a) $\mathcal{L}^{-1} \left(\frac{s}{(s^2+1)^2} \right) = \frac{t \sin t}{2}$,
 as in (c) above

b) $\frac{1}{(s^2+1)^2}$ suggests some combination of $\frac{d}{ds} \left(\frac{1}{s^2+1} \right)$ and $\frac{d}{ds} \left(\frac{s}{s^2+1} \right)$

$$\mathcal{L}\{t \cos t\} = -\frac{d}{ds} \left(\frac{s}{s^2+1} \right) = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}$$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \rightarrow \text{what we want}$$

$$\therefore \mathcal{L}^{-1} \left\{ \frac{t}{(s^2+1)^2} \right\} = \frac{1}{2} [\sin t - t \cos t]$$

3B-5

a) $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t) dt$
 $= \int_0^\infty e^{-(s-a)t} f(t) dt$
 $= F(s-a),$

since $F(s) = \int_0^\infty e^{-st} f(t) dt.$

b) $F(s) = \int_0^\infty e^{-st} f(t) dt$

Differentiating under the integral sign, with respect to s : \otimes

$$F'(s) = \int_0^\infty -t e^{-st} f(t) dt,$$

since t is a constant with respect to the differentiation;

$$\begin{aligned} &= \mathcal{L}\{-t f(t)\} \\ &= -\mathcal{L}\{t f(t)\}. \end{aligned}$$

 \otimes this is legal if $f(t)$ is continuous and of exponential order.**3B-6**

$$y'' + ty = 0, \quad y(0)=1, y'(0)=0$$

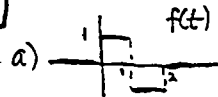
Take the Laplace transform:

$$(s^2 Y - s) - \frac{d}{ds} Y = 0$$

$$\frac{dY}{ds} = s^2 Y = -s,$$

(which is first order, linear).

3C-1



Using $u(t)$: $f(t) = u(t) - 2u(t-1) + u(t-2)$
 $\therefore F(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} = \frac{1}{s}(1 - 2e^{-s} + e^{-2s})$

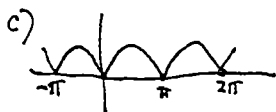
Directly:
 $F(s) = \int_0^1 e^{-st} dt - \int_1^2 e^{-st} dt = \frac{1}{s}(1 - e^{-s})^2$
 (by straight calc.)



Using $u(t)$: $f(t) = t^2 u(t) - u(t-1) - 2(t-1) + u(t-2)(t-2)$

$\therefore F(s) = \frac{1}{s^2}(1 - 2e^{-s} + e^{-2s})$

Directly:
 $F(s) = \int_0^1 t e^{-st} dt + \int_1^2 (2-t) e^{-st} dt$ [integrate each s by parts]
 $= \left[\frac{t e^{-st}}{-s} \right]_0^1 - \left[\frac{e^{-st}}{(-s)^2} \right]_0^1 + (2-t) \left[\frac{e^{-st}}{-s} \right]_1^2 - \left[\frac{e^{-st}}{(-s)^2} \right]_1^2$ (which agrees with * after canceling terms)



$|\sin t| = (-1)^n \sin t$,
 $n\pi \leq t \leq (n+1)\pi$.

This can be done directly, (adding up the integrals over even + odd intervals):

$F(s) = \int_0^\infty |\sin t| e^{-st} dt = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} (-1)^n \sin t \cdot e^{-st} dt$

Change variable: $u = t - n\pi$
 $= \sum_{n=0}^\infty \int_0^\pi (-1)^n \sin(u+n\pi) e^{-s(u+n\pi)} du$

$\sin(u+n\pi) = (-1)^n \sin u$; $e^{-sn\pi}$ is a "constant"

$= \sum_{n=0}^\infty e^{-sn\pi} \int_0^\pi \sin u \cdot e^{-su} du$
 call it "K". Then $K = \frac{1 + e^{-s\pi}}{1 + s^2}$ (from tables)

$= K \cdot \sum_0^\infty e^{-sn\pi}$; adding up this geometric series gives

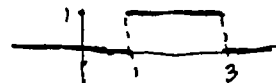
$= K \cdot \frac{1}{1 - e^{-s\pi}}$
 Ans: $\frac{1 + e^{-s\pi}}{(1 + s^2)(1 - e^{-s\pi})}$

3C-2

a) $\frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2}$ (partial fractions)
 $\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 3s + 2}\right\} = e^{-t} - e^{-2t} = f(t)$

$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2 + 3s + 2}\right\} = u(t-1)f(t-1)$
 $= u(t-1)(e^{-(t-1)} - e^{-2(t-1)})$

b) $\mathcal{L}^{-1}\left(\frac{e^{-s} - e^{-3s}}{s}\right) = \mathcal{L}^{-1}\left(\frac{e^{-s}}{s}\right) - \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s}\right)$
 $= u(t-1) - u(t-3)$



3C-3

a) $\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$
 $= \int_0^1 e^{-st} dt + \int_2^3 e^{-st} dt + \int_4^5 e^{-st} dt + \dots$
 $= \frac{e^0 - e^{-s}}{s} + \frac{e^{-2s} - e^{-3s}}{s} + \frac{e^{-4s} - e^{-5s}}{s} + \dots$
 $= \frac{1}{s} \cdot (e^0 - e^{-s} + e^{-2s} - e^{-3s} + \dots)$ (geometric series, whose sum is $\frac{1}{1 + e^{-s}}$)
 $= \frac{1}{s} \cdot \left(\frac{1}{1 + e^{-s}}\right)$

b) $f(t) = u(t) - u(t-1) + u(t-2) - \dots$
 $\therefore \mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \dots$
 $= \frac{1}{s} (e^0 - e^{-s} + e^{-2s} - e^{-3s} \dots)$
 $= \frac{1}{s} \cdot \frac{1}{1 + e^{-s}}$, as before.

3C-4

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{u(t-\pi) - u(t-2\pi)\}$$

$$= \frac{e^{-s\pi} - e^{-2s\pi}}{s}$$

The ODE is: $y'' + 2y' + 2y = h(t)$, $y(0)=0$, $y'(0)=1$

Laplace transform is:

$$(s^2 Y - 1) + 2(sY) + 2Y = \frac{e^{-s\pi} - e^{-2s\pi}}{s}$$

$$(s^2 + 2s + 2)Y = 1 + \frac{e^{-s\pi} - e^{-2s\pi}}{s}$$

$$Y = \frac{1}{(s+1)^2 + 1} \left[1 + \frac{e^{-s\pi} - e^{-2s\pi}}{s} \right]$$

By partial fractions

$$\frac{1}{(s^2 + 2s + 2)s} = \frac{-s/2 - 1}{s^2 + 2s + 2} + \frac{1/2}{s}$$

$$= \frac{-1/2(s+1) - 1/2}{(s+1)^2 + 1} + \frac{1/2}{s}$$

$$\therefore y = e^{-t} \sin t + \frac{1}{2} [1 - e^{-(t-\pi)} (\sin(t-\pi) + \cos(t-\pi))] u(t-\pi)$$

$$- \frac{1}{2} [1 - e^{-(t-2\pi)} (\sin(t-2\pi) + \cos(t-2\pi))] u(t-2\pi)$$

$$\therefore y = \begin{cases} e^{-t} \sin t, & (0 \leq t \leq \pi) \\ \frac{1}{2} (1 + \frac{e^\pi}{2}) e^{-t} \sin t + \frac{e^\pi}{2} e^{-t} \cos t, & (\pi \leq t \leq 2\pi) \\ (\frac{1 + e^\pi + e^{2\pi}}{2}) e^{-t} \sin t + (\frac{e^\pi}{2} + \frac{e^{2\pi}}{2}) e^{-t} \cos t, & (2\pi \leq t) \end{cases}$$

3C-5

$$\mathcal{L}\{u(t) \cdot t\} = \mathcal{L}\{t\} = \frac{1}{s^2}$$

$y'' - 3y' + 2y = r(t)$, $y(0)=1$, $y'(0)=0$ gives:

$$(s^2 Y - s) - 3(sY - 1) + 2Y = \frac{1}{s^2}$$

$$(s^2 - 3s + 2)Y = s - 3 + \frac{1}{s^2}$$

$$Y = \frac{s-3}{(s-2)(s-1)} + \frac{1}{s^2(s-2)(s-1)}$$

$$= \frac{s^3 - 3s^2 + 1}{s^2(s-2)(s-1)}$$

cont'd above

3C-5

21. (cont'd) By partial fractions

$$Y = \frac{1}{s-1} - \frac{3/4}{s-2} + \frac{3/4}{s} + \frac{1/2}{s^2}$$

$$\therefore y = e^t - \frac{3}{4} e^{2t} + \frac{3}{4} + \frac{t}{2}$$

3D-1

22. $y'' + 2y' + y = \delta(t) + u(t-1)$, $y(0)=0$, $y'(0)=1$

$$(s^2 Y - 1) + 2sY + Y = 1 + \frac{e^{-s}}{s}$$

$$(s^2 + 2s + 1)Y = 2 + \frac{e^{-s}}{s}$$

Divide, use part. fractions:

$$Y = \frac{2}{(s+1)^2} + e^{-s} \left[\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} \right]$$

$$y = 2te^{-t} + u(t-1) [1 - e^{-(t-1)} - (t-1)e^{-(t-1)}]$$

$$= 2te^{-t} + u(t-1) [1 - te^{1-t}]$$

$$\therefore y(t) = \begin{cases} 2te^{-t}, & 0 \leq t \leq 1 \\ 1 + (2-e)te^{-t}, & t \geq 1 \end{cases}$$

3D-2

23. $y'' + y = r(t)$, $y(0)=0$, $y'(0)=1$

$$r(t) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

$$= 1 - u(t-\pi)$$

$$\therefore \mathcal{L}\{r(t)\} = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

So $(s^2 Y - 1) + Y = \frac{1 - e^{-\pi s}}{s}$

$$Y = \frac{1}{s^2 + 1} + \frac{1}{s(s^2 + 1)} - \frac{e^{-\pi s}}{s(s^2 + 1)}$$

$$y = \sin t + 1 - \cos t$$

$$- (1 - \cos(t-\pi)) u(t-\pi)$$

[$= -\cos t$]

$$\therefore y = \begin{cases} 1 + \sin t - \cos t, & 0 \leq t \leq \pi \\ \sin t - 2 \cos t, & t \geq \pi \end{cases}$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

3D-3

$$a) F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

$$= \sum_{n=0}^{\infty} \int_{nc}^{(n+1)c} e^{-st} f(t) dt$$

Also:
SEE BELOW

[breaking $[0, \infty)$ up into the intervals $[nc, (n+1)c]$.

Change variable: $u = t - nc$

$$\int_{nc}^{(n+1)c} e^{-st} f(t) dt = \int_0^c e^{-s(u+nc)} f(u) du,$$

since $f(u+nc) = f(u)$.

Therefore our sum becomes:

$$F(s) = \sum_{n=0}^{\infty} e^{-snc} \underbrace{\int_0^c e^{-su} f(u) du}_{\text{call this } K}$$

$$= K \sum_{n=0}^{\infty} (e^{-sc})^n \leftarrow \text{a geometric series, whose sum is}$$

$$= K \cdot \frac{1}{1 - e^{-sc}}$$

$$\therefore F(s) = \frac{1}{1 - e^{-sc}} \cdot \int_0^c e^{-su} f(u) du$$

FOR A BETTER WAY, SEE NEXT PAGE

b) For problem 19, $c = 2$

$$\int_0^2 e^{-su} f(u) du = \int_0^1 e^{-su} du$$

$$= \frac{1 - e^{-s}}{s}$$


$$\therefore F(s) = \frac{1}{1 - e^{-2s}} \cdot \frac{1 - e^{-s}}{s}$$

$$= \frac{1}{s \cdot (1 + e^s)}, \text{ as before.}$$

c) using the "definition" of $\delta(t)$

$$\delta * f(t) = \int_0^t \delta(t-u) f(u) du = \int_0^t \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [u(t-u) - u(t-u_1 - \epsilon)] f(u) du$$

$$= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t [u(t-u_1) - u(t-u_1 - \epsilon)] f(u) du = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int_0^t f(u) du - \int_0^{t-\epsilon} f(u) du \right]$$

(SHADY!) $= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{t-\epsilon}^t f(u) du = f(t)$, since 

3D-4

$$a) \frac{s}{(s+1)(s^2+4)} = \frac{1}{s+1} \cdot \frac{s}{s^2+4}$$

$$\therefore \mathcal{L}^{-1}\left(\frac{s}{(s+1)(s^2+4)}\right) = e^{-t} * \cos 2t$$

$$= \int_0^t e^{-(t-u)} \cos 2u du$$

$$= e^{-t} \int_0^t e^u \cos 2u du$$

$$= e^{-t} \left[\frac{e^t}{5} (\cos 2t + 2 \sin 2t) - \frac{1}{5} \right]$$

$$= \frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t - \frac{1}{5} e^{-t}$$

$$b) \frac{1}{(s^2+1)^2} = \frac{1}{s^2+1} \cdot \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = \sin t * \sin t$$

$$= \int_0^t \sin(t-u) \cdot \sin u du$$

Easiest is to use a trig identity:

$$= \int_0^t \frac{1}{2} [\cos(t-2u) - \cos t] du$$

$$= \frac{\sin t}{2} - \frac{t}{2} \cos t$$

3D-5

$$a) f(t) \xrightarrow{\mathcal{L}} F(s), \delta(t) \xrightarrow{\mathcal{L}} 1$$

$$\mathcal{L}\{\delta * f\} = 1 \cdot F(s) = F(s)$$

$$\therefore \delta * f(t) = f(t) u(t) = f(t),$$

[THIS IS JUST FORMAL] since $f(t) = 0, t \leq 0$.

b) Using the definition of $*$:

$$\delta * f = \int_0^t \delta(t-u) f(u) du$$

$$= \int_{-\infty}^{\infty} \delta(t-u) f(u) du \left. \begin{array}{l} \text{since} \\ \delta(t-u) = 0 \\ \text{except if} \\ u = t \end{array} \right\}$$

$$= f(t) \int_{-\infty}^{\infty} \delta(t-u) du$$

$$= f(t) \cdot 1$$

3D-6

$$(f * g)(t) = \int_0^t f(t-u)g(u)du$$

let $x = t-u$ (change variable u to the var. x in the integral)
 $dx = -du$

limits:

when $u = 0, x = t$
 when $u = t, x = 0$ \therefore integral becomes:

$$= -\int_t^0 f(x)g(t-x)dx = \int_0^t g(t-x)f(x)dx$$

$$= (g * f)(t).$$

3D-7

Taking the Laplace transform:

$$s^2 Y + k^2 Y = R(s),$$

where $R(s) = \mathcal{L}\{r(t)\}$.

$$\therefore Y = \frac{R(s)}{s^2 + k^2} = \frac{1}{s^2 + k^2} \cdot R(s)$$

$$\therefore y = \frac{1}{k} \sin kt * r(t)$$

$$= \frac{1}{k} \int_0^t \sin k(t-u) \cdot r(u) du.$$

3D-8

$$y'' + ay' + by = r(t), \quad y(0) = 0$$

$$y'(0) = 0$$

$$s^2 Y + asY + bY = R(s)$$

$$\therefore Y = \frac{1}{s^2 + as + b} \cdot R(s)$$

$$\therefore y = g(t) * r(t), \quad \text{where } g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + as + b}\right\}$$

$$y = \int_0^t g(t-u)r(u)du.$$

To interpret $g(t)$, consider the ODE-IVP

$$y'' + ay' + by = 0, \quad y(0) = 0$$

$$y'(0) = 1$$

then $s^2 Y - 1 + asY + bY = 0$

so $Y = \frac{1}{s^2 + as + b}$

and $y = g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + as + b}\right\}$. Thus $g(t)$ may be interpreted as the soln to this IVP.

3D-8

(continued)

$g(t)$ may also be interpreted as the solution to

$$y'' + ay' + by = \delta(t),$$

$$y(0) = 0, \quad y'(0) = 0$$

since this leads to

$$s^2 Y + asY + bY = 1$$

$$\text{or } Y = \frac{1}{s^2 + as + b},$$

so that $y = g(t)$.

3D-3



we have:

$$u(t-c)f(t-c) + f_0(t) = u(t)f(t),$$

$$\text{where } f_0(t) = \begin{cases} f(t), & 0 \leq t < c \\ 0 & \text{elsewhere} \end{cases}$$

\therefore taking LT's:

$$e^{-cs} F(s) + \int_0^c e^{-st} f(t) dt = F(s).$$

Solve for $F(s)$:

$$F(s) = \frac{1}{1 - e^{-cs}} \int_0^c e^{-st} f(t) dt.$$

(see above for another interp. of g)



M.I.T. 18.03 Ordinary Differential Equations
18.03 Notes and Exercises

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