V2. Gradient Fields
and Exact Differentials

1. Criterion for gradient fields.

Let $\mathbf{F} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$ be a two-dimensional vector field, where M and N are continuous functions. There are three equivalent ways of saying that $\mathbf{F}$ is conservative, i.e., a gradient field:

$$(1) \qquad \mathbf{F} = \nabla f \;\Leftrightarrow\; \int_P^Q \mathbf{F}\cdot d\mathbf{r} \;\text{ is path-independent } \;\Leftrightarrow\; \oint_C \mathbf{F}\cdot d\mathbf{r} = 0 \;\text{ for any closed } C$$

Unfortunately, these equivalent formulations don't give us any effective way of deciding if a given field $\mathbf{F}$ is a conservative field or not. However, if we assume that $\mathbf{F}$ is not just continuous but is even continuously differentiable (meaning: M_x, M_y, N_x, N_y all exist and are continuous), then there is a simple and elegant criterion for deciding whether or not $\mathbf{F}$ is a gradient field in some region.

Criterion. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be continuously differentiable in a region D. Then, in D,

$$(2) \qquad\qquad\qquad \mathbf{F} = \nabla f \;\text{ for some } f(x,y) \quad\Rightarrow\quad M_y = N_x \;.$$

Proof. Since $\mathbf{F} = \nabla f$, this means

$$
\begin{aligned}
M &= f_x &\text{and}&& N &= f_y \;. &&\text{Therefore,}\\
M_y &= f_{xy} &\text{and}&& N_x &= f_{yx} \;.
\end{aligned}
$$

But since these two mixed partial derivatives are continuous (since M_y and N_x are, by hypothesis), a standard 18.02 theorem says they are equal. Thus $M_y = N_x$. $\qquad\square$

This theorem may be expressed in a slightly different form, if we define the scalar function called the **two-dimensional curl** of $\mathbf{F}$ by

$$(3) \qquad\qquad\qquad\qquad \text{curl } \mathbf{F} \;=\; N_x - M_y \;.$$

Then (2) becomes

$$(2') \qquad\qquad\qquad\qquad \mathbf{F} = \nabla f \quad\Rightarrow\quad \text{curl } \mathbf{F} = 0 \;.$$

This criterion allows us to test $\mathbf{F}$ to see if it is a gradient field. Naturally, we would also like to know that the converse is true: if curl $\mathbf{F} = 0$, then $\mathbf{F}$ is a gradient field. As we shall see, however, this requires some additional hypotheses on the region D. For now, we will assume D is the whole plane. Then we have

Converse to Criterion. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ be continuously differentiable for all x, y.

$$(4) \qquad M_y = N_x \text{ for all } x,y \quad\Rightarrow\quad \mathbf{F} = \nabla f \;\text{ for some differentiable } f \text{ and all } x,y.$$

The proof of (4) will be postponed until we have more technique. For now we will illustrate the use of the criterion and its converse.

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Example 1.   For which value(s), if any of the constants a, b will $axy\,\mathbf{i} + (x^2 + by)\,\mathbf{j}$ be a gradient field?

Solution.   The partial derivatives are continuous for all x, y and $M_y = ax$, $N_x = 2x$. Thus by (2) and (4), $\mathbf{F} = \nabla f \iff a = 2$; b is arbitrary.

Example 2. Are the fields $\mathbf{F} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{x^2 + y^2}$, $\mathbf{G} = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$ conservative?

Solution.   We have (the second line follows from the first by interchanging x and y):

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{-x^2 + y^2}{(x^2 + y^2)^2}; \qquad \frac{\partial}{\partial x}\left(\frac{y}{x^2 + y^2}\right) = \frac{2xy}{(x^2 + y^2)^2};$$

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) = \frac{x^2 - y^2}{(x^2 + y^2)^2}; \qquad \frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right) = \frac{2yx}{(x^2 + y^2)^2};$$

from this, we see immediately that

$$\frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right) = \frac{\partial}{\partial x}\left(\frac{y}{x^2 + y^2}\right); \qquad \frac{\partial}{\partial y}\left(-\frac{y}{x^2 + y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right);$$

the two equations in the last line show respectively that $\mathbf{F}$ and $\mathbf{G}$ satisfy the criterion (2). However, neither field is defined at $(0,0)$, so that the converse (4) is not applicable. So the question cannot be decided just on the basis of (2) and (4). In fact, it turns out that $\mathbf{F}$ is a gradient field, since one can check that

$$\mathbf{F} = \nabla \ln \sqrt{x^2 + y^2} = \nabla \ln r\,.$$

On the other hand, $\mathbf{G}$ is not conservative, since if C is the unit circle $x = \cos t$, $y = \sin t$, we have

$$\oint_C \mathbf{G} \cdot d\mathbf{r} = \oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} = \int_0^{2\pi} \frac{\sin^2 t\,dt + \cos^2 t\,dt}{\sin^2 t + \cos^2 t} = 2\pi \neq 0.$$

We will return later on in these notes to this example.

2.   Finding the potential function.

Example 2 above raises the question of how we found the function $\frac{1}{2}\ln(x^2 + y^2)$. More generally, if we know that $\mathbf{F} = \nabla f$ — for example if curl $\mathbf{F} = 0$ in the whole xy-plane — how do we find the function $f(x, y)$? There are two methods; some students prefer one, some the other.

Method 1.   Suppose $\mathbf{F} = \nabla f$. By the Fundamental Theorem for Line Integrals,

$$(5) \qquad \int_{(x_0,y_0)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r} = f(x, y) - f(x_0, y_0)\,.$$

Read from left to right, (5) gives us an easy way of finding the line integral in terms of $f(x, y)$. But read right to left, it gives us a way of finding $f(x, y)$ by using the line integral:

$$(5') \qquad f(x, y) = \int_{(x_0,y_0)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r} + c\,.$$

(Here c is an arbitrary constant of integration; as (5′) shows, $c = f(x_0, y_0)$.)

Remark.   It is common to refer to $f(x, y)$ as the (mathematical) **potential function**. The potential function used in physics is $-f(x, y)$. The negative sign is used by physicists so that the potential difference will represent work done *against* the field **F**, rather than work done *by* the field, as the convention is in mathematics.

**Example 3.**   Let $\mathbf{F} = (x + y^2)\,\mathbf{i} + (2xy + 3y^2)\,\mathbf{j}$. Verify that **F** satisfies the Criterion (2), and use method 1 above to find the potential function $f(x, y)$.

Solution.   We verify (2) immediately: $\dfrac{\partial(y^2)}{\partial y} \;=\; 2y \;=\; \dfrac{\partial(2xy)}{\partial x}$.

We use (5′). The point (x_0, y_0) can be any convenient starting point; $(0, 0)$ is the usual choice, if the integrand is defined there. (We will subscript the variables to avoid confusion with the variables of integration, but you don't have to.) By (5′),

(6) $$f(x_1, y_1) \;=\; \int_{(0,0)}^{(x_1, y_1)} (x + y^2)\, dx + (2xy + 3y^2)\, dy \;.$$

Since the integral is path-independent, we can choose any path we like. The usual choice is the one on the right, as it simplifies the computations. (Most of what follows you can do mentally, with a little practice.)
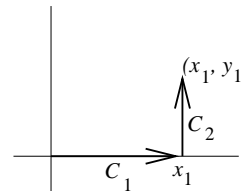
On C_1, we have $y = 0$, $dy = 0$, so the integral on C_1 becomes $\displaystyle\int_0^{x_1} x\, dx \;=\; \dfrac{1}{2}\, x_1^2$.

On C_2, we have $x = x_1$, $dx = 0$, so the integral is $\displaystyle\int_0^{y_1} (2x_1 y + 3y^2)\, dy \;=\; x_1 y_1^2 + y_1^3$.

Adding the integrals on C_1 and C_2 to get the integral along the entire path, and dropping the subscripts, we get by (6) and (5′)

$$f(x, y) = \frac{1}{2} x^2 + xy^2 + y^3 + c \;.$$

(The constant of integration is added by (5′), since the choice of starting point was arbitrary. You should always confirm the answer by checking that $\nabla f = \mathbf{F}$.)   □

Method 2.   Once again suppose $\mathbf{F} = \nabla f$, that is $M\,\mathbf{i} + N\,\mathbf{j} \;=\; f_x\,\mathbf{i} + f_y\,\mathbf{j}$. It follows that

(7) $$f_x = M \qquad \text{and} \qquad f_y = N \;.$$

These are two equations involving partial derivatives, which we can solve simultaneously by integration. We illustrate using the previous example: $\mathbf{F} = (x + y^2,\; 2xy + 3y^2)$.

Solution by Method 2.   Using the first equation in (7),

$$\frac{\partial f}{\partial x} \;=\; x + y^2. \qquad\qquad \text{Hold } y \text{ fixed, integrate with respect to } x\!:$$

(8) $$f \;=\; \frac{1}{2}\, x^2 + y^2 x + g(y). \qquad \text{where } g(y) \text{ is an arbitrary function of } y.$$

To find $g(y)$, we calculate $\dfrac{\partial f}{\partial y}$ two ways:

$$\frac{\partial f}{\partial y} = 2yx + g'(y) \qquad\qquad \text{by (8), while}$$

$$\frac{\partial f}{\partial y} = 2xy + 3y^2 \qquad\qquad \text{from (7), second equation.}$$

Comparing these two expressions, we see that $g'(y) = 3y^2$, so $g(y) = y^3 + c$. Putting it all together, using (8), we get $f(x,y) = \frac{1}{2}x^2 + y^2x + y^3 + c$, as before. $\square$

In the first method, the answer is written down immediately as a line integral; the rest of the work is in evaluating the integral, which goes quickly, since on a horizontal or vertical path either $dx = 0$ or $dy = 0$.

In the second method, the answer is obtained by an algorithm involving several steps which should be carried out in the right order.

The first method has the advantage of reminding you each time how $f(x,y)$ is defined and what it means, facts of theoretical and practical importance. The second method has the advantage of requiring no knowledge of line integrals, which makes it popular with students; on the other hand, when done in three dimensions, the bookkeeping gets more complicated, whereas in the first method it does not; overall, the first method is faster, provided you are confident enough to do some of the work mentally.

3.   Exact differentials.

The formal expressions $M(x,y)\,dx + N(x,y)\,dy$ which have appeared as the integrands in our line integrals are called **differentials**. In some applications, most notably thermodynamics, one usually works directly with the differential $M\,dx + N\,dy$ and its line integral $\int M\,dx + N\,dy$, without considering or using the associated vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$. Therefore it is important to have the results about gradient fields in this section translated into the language of differentials. We do this now.

If $f(x,y)$ is a differentiable function, its **total differential** df (or simply *differential*) is by definition the expression

$$(9) \qquad\qquad\qquad df = f_x dx + f_y dy \ .$$

For example, if $f(x,y) = x^2 y^3$, then $d(x^2 y^3) = 2xy^3 dx + 3x^2 y^2 dy$.

We call the differential $M\,dx + N\,dy$ **exact**, in a region D where M and N are defined, if it is the total differential of some function $f(x,y)$ in this region, i.e., if in D,

$$(10) \qquad\qquad M = f_x \qquad \text{and} \qquad N = f_y, \qquad \text{for some } f(x,y).$$

From this we see that the relation between differentials and vector fields is

$$M\,dx + N\,dy \quad\text{is exact} \quad \Leftrightarrow \quad M\,\mathbf{i} + N\,\mathbf{j} \quad\text{is a gradient field}$$
$$M\,dx + N\,dy = df \quad \Leftrightarrow \quad M\,\mathbf{i} + N\,\mathbf{j} = \nabla f \ .$$

In this language, the criterion (2) and its partial converse (4) become the

Exactness Criterion.   *Assume M and N are continuously differentiable in a region D of the plane. Then in this region,*

(11) $M\,dx + N\,dy$ exact $\Rightarrow$ $M_y = N_x$;

(12)   *if D is the whole xy-plane,* $M_y = N_x$ $\Rightarrow$ $M\,dx + N\,dy$ exact.

If the exactness criterion shows that $M\,dx + N\,dy$ is exact, then the function $f(x, y)$ may be found by either of the two methods previously described.

Exercises: Section 4C.

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