

## V14. Some Topological Questions

We consider once again the criterion for a gradient field. We know that

$$(1) \quad \mathbf{F} = \nabla f \quad \Rightarrow \quad \text{curl } \mathbf{F} = \mathbf{0} ,$$

and inquire about the converse. It is natural to try to prove that

$$(2) \quad \text{curl } \mathbf{F} = \mathbf{0} \quad \Rightarrow \quad \mathbf{F} = \nabla f$$

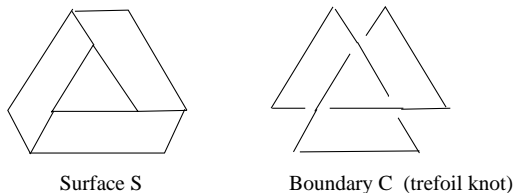
by using Stokes' theorem: if  $\text{curl } \mathbf{F} = \mathbf{0}$ , then for any closed curve  $C$  in space,

$$(3) \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0.$$

The difficulty is that we are given  $C$ , but not  $S$ . So we have to ask:

**Question.** Let  $D$  be a region of space in which  $\mathbf{F}$  is continuously differentiable. Given a closed curve lying in  $D$ ; is it the boundary of some two-sided surface  $S$  lying inside  $D$ ?

We explain the “two-sided” condition. Some surfaces are only one-sided: if you start painting them, you can use only one color, if you don't allow abrupt color changes. An example is  $S$  below, formed giving three half-twists to a long strip of paper before joining the ends together.



$S$  has only one side. This means that it cannot be oriented: there is no continuous choice for the normal vector  $\mathbf{n}$  over this surface. (If you start with a given  $\mathbf{n}$  and make it vary continuously, when you return to the same spot after having gone all the way around, you will end up with  $-\mathbf{n}$ , the oppositely pointing vector.) For such surfaces, it makes no sense to speak of “the flux through  $S$ ”, because there is no consistent way of deciding on the positive direction for flow through the surface. Stokes' theorem does not apply to such surfaces.

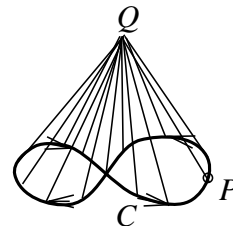
To see what practical difficulties this causes, even when the domain is all of 3-space, consider the curve  $C$  in the above picture. It's called the trefoil knot. We know it is the boundary of the one-sided surface  $S$ , but this is no good for equation (3), which requires that we find a *two-sided* surface which has  $C$  for boundary.

There are such surfaces; try to find one. It should be smooth and not cross itself. If successful, consider yourself a brown-belt topologist.

The preceding gives some ideas about the difficulties involved in finding a two-sided surface whose boundary is a closed curve  $C$  when the curve is knotted, i.e., cannot be continuously deformed into a circle without crossing itself at some point during the deformation. It is by no means clear that such a two-sided surface exists in general.

There are two ways out of the dilemma.

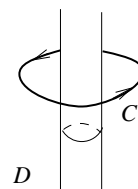
1. If we allow the surface to cross itself, and allow it to be not smooth along some lines, we can easily find such a two-sided surface whose boundary is a given closed curve  $C$ . The procedure is simple. Pick some fixed point  $Q$  not on the curve  $C$ , and join it to a point  $P$  on the curve (see the figure). Then as  $P$  moves around  $C$ , the line segment  $QP$  traces out a surface whose boundary is  $C$ . It will not be smooth at  $Q$ , and it will cross itself along a certain number of lines, but it's easy to see that this is a two-sided surface.



The point now is that Stokes' theorem can still be applied to such a surface: just use subdivision. Divide up the surface into skinny "triangles", each having one vertex at  $Q$ , and include among the edges of these triangles the lines where the surface crosses itself. Apply Stokes' theorem to each triangle, and add up the resulting equations.

2. Though the above is good enough for our purposes, it's an amazing fact that for any  $C$  there is always a smooth two-sided surface which doesn't cross itself, and whose boundary is  $C$ . (This was first proved around 1930 by van Kampen.)

The above at least answers our question affirmatively when  $D$  is all of 3-space. Suppose however that it isn't. If for instance  $D$  is the exterior of the cylinder  $x^2 + y^2 = 1$ , then it is intuitively clear that a circle  $C$  around the outside of this cylinder isn't the boundary of any finite surface lying entirely inside  $D$ .



A class of domains for which it is true however are the *simply-connected* ones.

**Definition.** A domain  $D$  in 3-space is **simply-connected** if each closed curve in it can be shrunk to a point without ever getting outside of  $D$  during the shrinking.

For example, 3-space itself is simply-connected, as is the interior or the exterior of a sphere. However the interior of a torus (a bagel, for instance) is not simply-connected, since any circle in it going around the hole cannot be shrunk to a point while staying inside the torus.

If  $D$  is simply-connected, then any closed curve  $C$  is the boundary of a two-sided surface (which may cross itself) lying entirely inside  $D$ . We can't prove this here, but it gives us the tool we need to establish the converse to the criterion for gradient fields in 3-space.

**Theorem.** Let  $D$  be a simply-connected region in 3-space, and suppose that the vector field  $\mathbf{F}$  is continuously differentiable in  $D$ . Then in  $D$ ,

$$(5) \quad \text{curl } \mathbf{F} = \mathbf{0} \quad \Rightarrow \quad \mathbf{F} = \nabla f .$$

**Proof.** According to the two fundamental theorems of calculus for line integrals (section V11.3), it is enough to prove that  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$  for every closed curve  $C$  lying in  $D$ .

Since  $D$  is simply-connected, given such a curve  $C$ , we can find a two-sided surface  $S$  lying entirely in  $D$  and having  $C$  as its boundary. Applying Stokes' theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0,$$

which shows that  $\mathbf{F}$  is conservative, and hence that  $\mathbf{F}$  is a gradient field.  $\square$

Summarizing, we can say that if  $D$  is simply-connected, the following statements are equivalent—if one is true, so are the other two:

$$(6) \quad \mathbf{F} = \nabla f \Leftrightarrow \operatorname{curl} \mathbf{F} = \mathbf{0} \Leftrightarrow \int_P^Q \mathbf{F} \cdot d\mathbf{r} \text{ is path independent.}$$

### Concluding remarks about Stokes' theorem.

Just as problems of sources and sinks lead one to consider Green's theorem in the plane for regions which are not simply-connected, it is important to consider such domains in connection with Stokes' theorem.

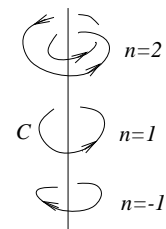
For example, if we put a closed loop of wire in space, the exterior of this loop — the region consisting of 3-space with the wire removed — is not simply-connected. If the wire carries current, the resulting electromagnetic force field  $\mathbf{F}$  will satisfy  $\operatorname{curl} \mathbf{F} = \mathbf{0}$ , but  $\mathbf{F}$  will not be conservative. In particular, the value of  $\oint \mathbf{F} \cdot d\mathbf{r}$  around a closed path which links with the loop will *not* in general be zero, (which explains why you can get power from a wire carrying current, even though the curl of its electromagnetic field is zero).

As an example, consider the vector field in 3-space

$$\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{r^2}, \quad r = \sqrt{x^2 + y^2}.$$

The domain of definition is  $xyz$ -space, with the  $z$ -axis removed (since the  $z$ -axis is where  $r = 0$ ). Just as before, (Section V2,p.2), we can calculate that  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$  if  $C$  is a suitably directed circle lying in a plane  $z = z_0$  and centered on the  $z$ -axis.

**Exercise.** By using Stokes' theorem for a surface with more than one boundary curve, show informally that for the field above,  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$  for any closed curve  $C$  going once around the  $z$ -axis, oriented so when that when the right thumb points in the direction  $\mathbf{k}$ , the fingers curl in the positive direction on  $C$ . Then show that if  $C$  goes around  $n$  times, the value of the integral will be  $2n\pi$ .



Suppose now the wire is a closed curve that is *knotted*, i.e., it cannot be continuously deformed to a circle, without crossing itself at some point in the deformation. Let  $D$  be the exterior of the wire loop, and consider the value of  $\oint_C \mathbf{F} \cdot d\mathbf{r}$  for a vector field  $\mathbf{F}$  defined in  $D$  and having  $\operatorname{curl} \mathbf{F} = \mathbf{0}$ . If one closed curve  $C_1$  can be deformed into another closed curve  $C_2$  without leaving  $D$  (i.e., without crossing the wire), then by using Stokes' theorem for surfaces with two boundary curves, we conclude

$$(6) \quad \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

More generally, two closed curves  $C_1$  and  $C_2$  are called *homologous*, written  $C_1 \sim C_2$ , if  $C_1$  and  $C_2'$  (this means  $C_2$  with its direction reversed) form the complete boundary of some surface lying entirely in  $D$ . Then by an extended form of Stokes' theorem, (6) will be true whenever  $C_1 \sim C_2$ . Thus the problem of determining the possible values for the line integral is reduced to the purely topological problem of finding a set of closed curves in  $D$ , no two of which are homologous, but such that every other closed curve is homologous to one of

the curves in the set. For any particular knot in 3-space, such a set can be determined by an algorithm, but if one asks for general results relating the appearance of the knot to the number of such basic curves that will be needed, one gets into unsolved problems of topology.

In another vein, the theorems of Green, Stokes, and Gauss (as the divergence theorem is often called) all relate an integral over the interior of some closed curve or surface with an integral over its boundary. There is a much more general result — the generalized Stokes' theorem — which connects an integral over an  $n$ -dimensional hypersurface with an integral taken over its  $n - 1$ -dimensional boundary. Green's and Stokes' theorems are the case  $n = 2$  of this result, while the divergence theorem is closely related to the case  $n = 3$  in 3-space. Just as the theorems we have studied are the key to an understanding of geometry and analysis in the plane and space, so this theorem is central to an understanding of  $n$ -dimensional space, and more general sorts of spaces called “ $n$ -dimensional manifolds”.

### Exercises: Section 6G

**18.02 Notes and Exercises by A. Mattuck and  
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