

X. EXPONENTIALS AND LOGARITHMS

1. The Exponential and Logarithm Functions.

We have so far worked with the algebraic functions — those involving polynomials and root extractions — and with the trigonometric functions. We now have to add to our list the exponential and logarithm functions, since these are used in your science and engineering courses from the beginning. Your book will handle the calculus of these functions; here we want to review briefly their algebraic properties, and look at applications; one or two of them might be new to you.

Where does one encounter exponentials and logarithms? In general, **exponentials** are used to express all sorts of simple growth and decay processes.

1. The growth of a **bacteria colony** which doubles in size every day: $y = y_0 2^t$, where t = time in days, y = population size, y_0 = the initial size, i.e., size at $t = 0$.
2. Dollars in a **bank account**, at 5% interest compounded annually: $A = A_0(1.05)^n$, where n = number of years, A = amount, A_0 = initial amount (the “principal”).
3. Amount of **radioactive substance**, with a 1 year half-life: $x = x_0(1/2)^n = x_0 2^{-n}$, where n = number of years, x = amount, x_0 = initial amount.

There are many other examples: the decay of electric charge on a capacitor, the way a hot and cold body come to the same temperature when they are brought together, the falling of a body through a resisting medium (a steel ball dropped into oil, for instance) — all involve exponentials when you express them in mathematical terms.

We use **logarithms** when the base of the exponential is unimportant and we want to focus our attention on the exponents instead. Suppose the base is 10; writing simply “log” for “log₁₀” in what follows, we have

$$y = 10^x \quad \Leftrightarrow \quad \log y = x.$$

1. **Star magnitude.** The observed brightness B of a star is described by comparing it to a standard brightness B_0 , using the equation

$$B = B_0 10^{-2m/5}.$$

The important thing is the number m called the *magnitude*; it is defined by the above equation, or equivalently, taking logs, by

$$m = -\frac{5}{2} \log \frac{B}{B_0}.$$

Thus, the higher the magnitude, the fainter the star. A star of brightness B_0 , if one existed, would have magnitude zero. Thus the constant B_0 gives the magnitude scale a zero point. This point was chosen so that the brightest stars (other than the sun) — Sirius and Vega, for example — would have magnitude near 1, i.e., be stars of the first magnitude.

2. **Orders of magnitude.** We speak of a kilometer as “three orders of magnitude” greater than a meter. That is

$$\frac{\text{km}}{\text{meter}} = 10^3, \quad 3 = \log\left(\frac{\text{km}}{\text{meter}}\right).$$

In general, when we compare two quantities A and B ,

$$\text{number of orders of magnitude by which } A \text{ is greater than } B = \log(A/B).$$

3. **Chemical pH.** In chemistry, the acidity or alkalinity of a solution is measured by the concentration $[\text{H}^+]$ of hydrogen ions in the solution, in units of moles/liter.

If for some solution the concentration $[\text{H}^+] = 10^n$, then we define

$$\text{pH of solution} = -n = -\log[\text{H}^+].$$

The negative sign is used to avoid always having negative pH's, since the concentration $[\text{H}^+]$ is very small.

The algebraic laws.

We will begin with a brief review of the algebra of exponentials and logarithms. Please study the examples given of calculations, since many students seem to have trouble with this.

We fix a number $a > 0$, called the **base**. The pair of equations

$$(1) \quad y = a^x \quad \log_a y = x$$

are equivalent; both express the same relation between x and y . If you substitute one equation into the other, you get two other equations which are often more useful in practice than those in (1) (the examples will illustrate):

$$(2) \quad y = a^{\log_a y} \quad \log_a(a^x) = x.$$

In the first, we take the log then exponentiate; in the second, we exponentiate, then take the log; in both cases, we end up where we started.

The two basic algebraic laws governing exponentials and logarithms are

$$(3) \quad a^{x_1+x_2} = a^{x_1}a^{x_2} \quad \log_a y_1 + \log_a y_2 = \log_a y_1 y_2$$

$$(4) \quad (a^{x_1})^k = a^{kx_1} \quad \log_a y_1^k = k \log_a y_1$$

The laws on either side follow from those on the other side by substituting $y_i = a^{x_i}$ or $x_i = \log_a y_i$, and then using (1) or (2).

Graphs of the exponential and log functions

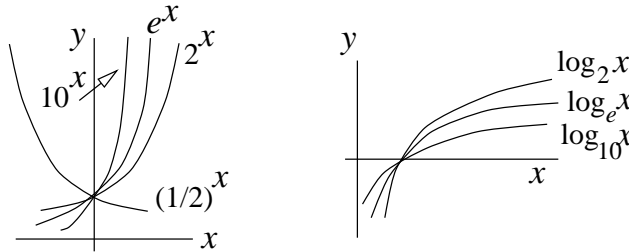
The picture on the left below shows some graphs of the exponential function a^x for different choices of bases. (We will discuss the base $e \approx 2.718$ presently.)

The graph rises if $a > 1$, falls if $a < 1$. Since $\left(\frac{1}{a}\right)^x = a^{-x}$, the graph of $(1/a)^x$ is obtained by reflecting the graph of a^x in the y -axis. (See Notes G, p. G.2.)

The pair of equivalent equations $y = a^x$ and $x = \log_a y$ show that a^x and $\log_a x$ are *inverse functions*, according to Notes G, pp. G.8,9. Thus the graph of $\log_a x$ is obtained by reflecting the graph of a^x in the diagonal line $y = x$. The reasoning for this needs a lot of repetition to sink in, so we repeat it briefly once more. Namely, consider in turn the three equations

$$y = a^x \qquad x = a^y \qquad y = \log_a x$$

Since the second equation arises from the first by switching x and y , the graph of the second equation is obtained by flipping the first graph around the diagonal line $y = x$; the third equation has the same graph as the second, since it's the same equation — we've only solved it for y . To sum up: the first and third equations have graphs which are reflections of each other about $y = x$.



The base e . Changing the base.

There are really only two or three bases in common use.

For areas in science and engineering where calculus does not enter, or where at least historically it was not used, the base 10 is the usual choice, because it associates naturally with the decimal system, making rough estimations of the size of exponentials a fairly easy thing to do.

On the other hand, if you are going to have to differentiate or integrate, the base $e \approx 2.718$ is the usual choice. (The symbol e was introduced by Euler — the first letter of both “exponential” and “Euler”.) As you will see, it is the base which makes for the simplest-looking differentiation and integration formulas for exponentials and logs.

(To the above two, one should perhaps add the base 2, which is useful in computer science, with its bits and bytes.)

For base e and base 10, the corresponding notations for the logarithm are

$$\log_e x = \ln x \quad (\text{the “natural logarithm”}); \qquad \log_{10} x = \log x.$$

(Warning: Many pure mathematics books use $\log x$ for the natural logarithm; science and engineering books generally use $\ln x$. Some computer science books use $\log x$ to mean $\log_2 x$.)

Using science and engineering notation, the logarithm laws read, for the bases e and 10 :

$$(5) \quad \begin{array}{ll} \ln(ab) = \ln a + \ln b & \log(ab) = \log a + \log b \\ \ln(a^b) = b \ln a & \log(a^b) = b \log a \end{array}$$

and

$$(6) \quad \begin{array}{ll} \ln(e^x) = x, & e^{\ln x} = x & \log(10^x) = x, & 10^{\log x} = x \\ \ln e = 1, \ln 1 = 0; & e^1 = e, e^0 = 1 & \log 10 = 1, \log 1 = 0; & 10^1 = 10, 10^0 = 1. \end{array}$$

Other bases. The reason why it is not necessary to use other bases is that all other exponentials and logarithms can be written in terms of e^x and $\ln x$, or if one prefers, in terms of 10^x and $\log x$. Since in calculus we use the base e , let's use this base. Using (6), we have for any base $a > 0$,

$$a^x = (e^{\ln a})^x = e^{x \ln a}.$$

Thus the functions e^{kx} , for different constants k , represent all of the exponential functions a^x . no matter what the base a is.

As for logarithms, starting with the equations

$$a^x = y, \quad x = \log_a y,$$

taking the natural log of the equation on the left gives

$$x(\ln a) = \ln y.$$

Solving for x and substituting into the second equation, we get

$$\log_a y = \frac{\ln y}{\ln a}.$$

Finally, changing the variable from y to x , we have

$$(7) \quad \log_a x = \frac{\ln x}{\ln a}; \quad \text{in particular,} \quad \log x = \frac{\ln x}{\ln 10}.$$

Study these transformations until you can reproduce them easily.

In calculations, students usually find it easy to use the laws of exponentials and logs. The chief trouble is in passing from the log to the exponential and vice-versa. Study the following examples, and don't go further until you have tried some of the exercises. Notice that you can almost always get through using (2), whereas the equations (1) sometimes aren't good enough, and even when they are, they require too much thinking. The general principle which governs the following calculations is:

$$(8) \quad \textit{To get rid of logs, exponentiate; to get rid of exponentials, take the logs.}$$

Example 1. If one liquid has a pH which is 4 greater than the pH of a second liquid, what is the relation between the $[\text{H}^+]$ concentrations of the two liquids? (cf. p.X.1)

First Solution. Using subscripts to distinguish the two liquids,

$$\begin{aligned}(\text{pH})_1 &= (\text{pH})_2 + 4 \\ -\log [\text{H}^+]_1 &= -\log [\text{H}^+] + 4\end{aligned}$$

Multiplying through by -1 , then exponentiating both sides (base 10) and using (6),

$$[\text{H}^+]_1 = [\text{H}^+]_2 \cdot 10^{-4}.$$

Second Solution. Starting from the second line, put both logs on the left side, and combine them using (5) (remember that $-\log a = \ln(1/a)$), then exponentiate both sides (or use (1)).

Example 2 A radioactive substance decays according to the law $y = y_0 e^{-kt}$, $k > 0$. Two measurements are made: at time t_i , there is an amount y_i present, where $i = 1, 2$. Express k in terms of the y_i and t_i .

Solution. We have $y_2 = y_0 e^{-kt_2}$ and $y_1 = y_0 e^{-kt_1}$. Dividing one equation by the other to get rid of y_0 ,

$$\begin{aligned}y_2/y_1 &= e^{-kt_2}/e^{-kt_1} = e^{-k(t_2-t_1)}, \\ \ln(y_2/y_1) &= -k(t_2 - t_1), \\ k &= -\frac{\ln(y_2/y_1)}{t_2 - t_1} = -\frac{\ln y_2 - \ln y_1}{t_2 - t_1}.\end{aligned}$$

Example 3. Solve for y the equation $\ln(y+1) = \ln y + x - 2$.

Solution. Exponentiating both sides, $y+1 = y \cdot e^{x-2}$; this is of the form $y+1 = Ay$, whose solution for y is

$$y = \frac{1}{1-A} = \frac{1}{1-e^{x-2}}.$$

Example 4. Solve for y : $(1+x)^3 = 4^{-y^2}$.

Solution. Take the log of both sides, to get the y downstairs:

$$\begin{aligned}3 \ln(1+x) &= -y^2 \ln 4 \\ \left[-\frac{3}{\ln 4} \ln(1+x) \right]^{1/2} &= y.\end{aligned}$$

Exercises: Section 1H