Unit 6. Additional Topics

6A. Indeterminate forms; L'Hospital's rule

6A-1 a) $\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} \frac{3\cos 3x}{1} = 3$

b) $\lim_{x\to 0} \frac{\cos(x/2) - 1}{x^2} = \lim_{x\to 0} \frac{(-1/2)\sin(x/2)}{2x} = \lim_{x\to 0} \frac{(-1/4)\cos(x/2)}{2} = -1/8$

c) $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$

d) $\lim_{x\to 0} \frac{x^2 - 3x - 4}{x+1} = -4.$ Can't use L'Hospital's rule.

e) $\lim_{x\to 0} \frac{\tan^{-1} x}{5x} = \lim_{x\to 0} \frac{1/(1+x^2)}{5} = 1/5$

f) $\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1 - \cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = 1/6$

g) $\lim_{x\to 1} \frac{x^a - 1}{x^b - 1} = \lim_{x\to 1} \frac{ax^{a-1}}{bx^{b-1}} = a/b$

h) $\lim_{x\to 1} \frac{\tan(x)}{\sin(3x)} = \frac{\tan 1}{\sin 3}.$ Can't use L'Hospital's rule.

i) $\lim_{x\to\pi} \frac{\ln\sin(x/2)}{x - \pi} = \lim_{x\to\pi} \frac{(1/2)\cot(x/2)}{1} = 0$

j) $\lim_{x\to\pi} \frac{\ln\sin(x/2)}{(x-\pi)^2} = \lim_{x\to\pi} \frac{(1/2)\cot(x/2)}{2(x-\pi)} = \lim_{x\to\pi} \frac{(-1/4)\csc^2(x/2)}{2} = -1/8$

6A-2 a) $x^x = e^{x\ln x} \to e^0 = 1$ as $x \to 0^+$ because

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$$

b) $x^{1/x} \to 0$ as $x \to 0^+$ because $x \to 0$ and $1/x \to \infty$.

Slow way using logs:

$x^{1/x} = e^{\frac{\ln x}{x}} \to e^{-\infty} = 0$ as $x \to 0^+$ because

$\lim_{x\to 0^+} \frac{\ln x}{x} = \frac{-\infty}{0^+} = -\infty.$ (Can't use L'Hospital's rule.)

c) Can't use L'Hospital's rule. Here are two ways:

$(1/x)^{\ln x} \to (\infty)^{-\infty} = 0$ or $(1/x)^{\ln x} = e^{\ln x \ln(1/x)} = e^{-(\ln x)^2} \to e^{-\infty} = 0$

d) $(\cos x)^{1/x} = e^{\frac{\ln\cos x}{x}} \to e^0 = 1$ as $x \to 0^+$ because

$$\lim_{x\to 0^+} \frac{\ln\cos x}{x} = \lim_{x\to 0^+} \frac{-\tan x}{1} = 0$$

e) $x^{1/x} = e^{\frac{\ln x}{x}} \to e^0 = 1$ as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

f) $(1+x^2)^{1/x} = e^{\frac{\ln(1+x^2)}{x}} \to e^0 = 1$ as $x \to 0^+$ because

$$\lim_{x\to 0^+} \frac{\ln(1+x^2)}{x} = \lim_{x\to 0^+} \frac{2x/(1+x^2)}{1} = 0$$

g) $(1+3x)^{10/x} = e^{\frac{10\ln(1+3x)}{x}} \to e^{30}$ as $x \to 0^+$ because

$$\lim_{x\to 0^+} \frac{10\ln(1+3x)}{x} = \lim_{x\to 0^+} \frac{10\cdot 3/(1+3x)}{1} = 30$$

h) $\lim_{x\to\infty} \frac{x+\cos x}{x} = (?) \lim_{x\to\infty} \frac{1-\sin x}{1}$ But the second limit does not exist, so L'Hospital's rule is **inconclusive**. But the first limit does exist after all:

$$\lim_{x\to\infty} \frac{x+\cos x}{x} = \lim_{x\to\infty} 1 + \frac{\cos x}{x} = 1$$

because

$$\frac{|\cos x|}{x} \le \frac{1}{x} \to 0 \quad \text{as } x\to\infty$$

Commentary: L'Hospital's rule does a poor job with oscillatory functions.

i) Fast way: Substitute $u = 1/x$.

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{u\to 0} \frac{\sin u}{u} = \lim_{u\to 0} \frac{\cos u}{1} = 1$$

Slower way:

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \frac{(-1/x^2)\cos(1/x)}{-1/x^2} = \cos 0 = 1$$

j) $\left(\frac{x}{\sin x}\right)^{1/x^2} = e^{\frac{\ln(x/\sin x)}{x^2}} \to e^{\frac{1}{6}}$ because

$$\lim_{x\to 0^+} \frac{\ln(x/\sin x)}{x^2} = 1/6$$

This is a difficult limit. Although it can be done by L'Hospital's rule the easiest way to work it out is with quadratic (and even cubic!) approximations:

$$\frac{x}{\sin x} \approx \frac{x}{x - x^3/6} = \frac{1}{1 - x^2/6} \approx 1 + x^2/6$$

Hence,

$$\ln(x/\sin x) \approx \ln(1 + x^2/6) \approx x^2/6$$

Therefore,

$$\frac{1}{x^2}\ln(x/\sin x) \to 1/6 \quad \text{as } x \to 0$$

k) Obvious cases: If the exponents are positive (or one 0 and the other positive) then the limit is infinite. If the exponents are both negative (or one 0 and the other negative) then the limit is 0. Also if both exponents are 0 the limit is 1. *(continued →)*

The remaining cases are the ones where a and b have opposite sign. In both cases a wins. In other words, $a < 0$ implies the limit is 0 and $a > 0$ implies the limit is ∞. To show this requires only one use of L'Hospital's rule. For $\alpha > 0$,

$$\lim_{x\to\infty} \frac{x^\alpha}{\ln x} = \lim_{x\to\infty} \frac{\alpha x^{\alpha-1}}{1/x} = \lim_{x\to\infty} \alpha x^\alpha = \infty$$

If $a > 0$ and $b < 0$, let $c = -b > 0$. Then

$$x^a(\ln x)^b = \left(\frac{x^{a/c}}{\ln x}\right)^c \to \infty \quad \text{as } x \to \infty$$

using $\alpha = a/c > 0$. The case $a < 0$ and $b > 0$ is the reciprocal so it tends to 0.

6A-3 Using L'Hospital's rule and $\dfrac{d}{da}x^{a+1} = x^{a+1}\ln x$,

$$\lim_{a\to-1}\left(\frac{x^{a+1}}{a+1} - \frac{1}{a+1}\right) = \lim_{a\to-1}\frac{x^{a+1}-1}{a+1} = \lim_{a\to-1}\frac{x^{a+1}\ln x}{1} = \ln x$$

6A-4

$$\int_1^x t^a \ln t dt = \frac{x^{a+1}\ln x}{a+1} - \frac{x^{a+1}}{(a+1)^2} + \frac{1}{(a+1)^2}$$

Therefore, using L'Hospital's rule and $\dfrac{d}{da}x^{a+1} = x^{a+1}\ln x$,

$$\begin{aligned}\lim_{a\to-1}\int_1^x t^a \ln t dt &= \lim_{a\to-1}\frac{(a+1)x^{a+1}\ln x - x^{a+1} + 1}{(a+1)^2} \\ &= \lim_{a\to-1}\frac{(a+1)x^{a+1}(\ln x)^2}{2(a+1)} \\ &= (\ln x)^2/2 = \int_1^x t^{-1}\ln t dt\end{aligned}$$

6A-5 You can't use L'Hospital's rule for $\lim_{x\to 0}\dfrac{6x-4}{2-2x}$ because the nominator and denominator are not going to zero as $x \to 0$. The first equality is true, but the second one is false.

6A-6 a) $y = xe^{-x}$ is defined on $-\infty < x < \infty$.

$$y' = (1-x)e^{-x} \text{ and } y'' = (-2+x)e^{-x}$$

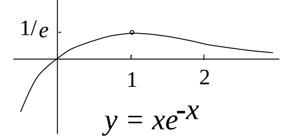
Therefore, $y' > 0$ for $x < 1$ and $y' < 0$ for $x > 1$; $y'' > 0$ for $x > 2$ and $y'' < 0$ for $x < 2$.

Endpoint values: $y \to -\infty$ as $x \to -\infty$, because $e^{-x} \to \infty$ as $x \to -\infty$. By L'Hospital's rule,

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0$$

Critical value: $y(1) = 1/e$.

Graph: $(-\infty, -\infty) \nearrow (1, 1/e) \searrow (\infty, 0)$.



Concave up on: $2 < x < \infty$, concave down on: $-\infty < x < 2$.

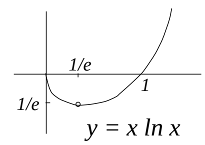
b) $y = x \ln x$ is defined on $0 < x < \infty$.

$$y' = \ln x + 1, \quad y'' = 1/x$$

Therefore, $y' > 0$ for $x > 1/e$ and $y' < 0$ for $x < 1/e$; $y'' > 0$ for all $x > 0$.

Endpoint values: As $x \to \infty$, both x and $\ln x$ tend to infinity, so $y \to \infty$. By L'Hospital's rule,

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{x} = \lim_{x \to 0^+} \frac{1/x}{1} = 0$$



Critical value: $y(1/e) = -1/e$.

Graph: $(0,0) \searrow (1/e, -1/e) \nearrow (\infty, \infty)$, crossing zero at $x = e$. Concave up for all $x > 0$.

c) $y = x/\ln x$ is defined on $0 < x < \infty$, except for $x = 1$.

$$y' = \frac{\ln x - 1}{(\ln x)^2}$$

Thus, $y' < 0$ for $0 < x < 1$ and for $1 < x < e$ and $y' > 0$ for $x > e$;

Endpoint values: $y \to 0$ as $x \to 0^+$ because $x \to 0$ and $1/\ln x \to 0$. L'Hôpital's rule implies

$$\lim_{x \to \infty} \frac{x}{\ln x} = \lim_{x \to \infty} \frac{1}{1/x} = \infty$$

Singular values: $y(1^+) = \infty$ and $y(1^-) = -\infty$.

Critical value: $y(e) = e$.

Graph: $(0,0) \searrow (1, -\infty) \uparrow (1, \infty) \searrow (e, e) \nearrow (\infty, \infty)$.

To determine where it is convex and concave:

$$y'' = \frac{2 - \ln x}{x(\ln x)^3}$$

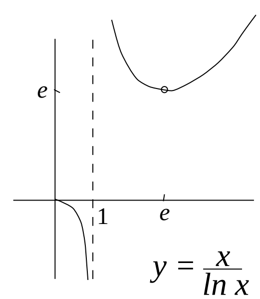
We have $y'' = 0$ when $\ln x = 2$, i.e., when $x = e^2$. From this,

$y'' < 0$ for $0 < x < 1$ and for $x > e^2$ and $y'' > 0$ for $1 < x < e^2$.

Concave (down) on: $0 < x < 1$ and $x > e^2$

Convex (concave up) on: $1 < x < e^2$

Inflection point: $(e^2, e^2/2)$ (too far to the right to show on the graph)



6B. Improper integrals

6B-1 $\dfrac{dx}{\sqrt{x^3+5}} < \dfrac{1}{\sqrt{x^3}}$ for $x > 0$

$\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^3+5}} < \int_1^\infty \frac{dx}{x^{3/2}}$ which converges, by INT (4)

Answer: converges

6B-2 $\dfrac{x^2dx}{x^3+2} \simeq \dfrac{1}{x}$ if $x >> 1$, so we guess divergence.

$\dfrac{x^2dx}{x^3+2} > \dfrac{1}{2x}$ if $2x^3 > x^3+2$ or $x^3 > 2$ or $x > 2^{1/3}$

$\displaystyle\int_2^\infty \frac{x^2dx}{x^3+2} > \frac{1}{2}\int_2^\infty \frac{dx}{x}$, which diverges by INT (4).

$\displaystyle\int_2^\infty \frac{x^2dx}{x^3+2}$ diverges, by comp.test, and so does $\displaystyle\int_0^\infty \frac{x^2dx}{x^3+2}$ by INT (3).

6B-3 $\displaystyle\int_0^1 \frac{dx}{x^3+x^2}$ integrand blows up at $x = 0$

$\dfrac{1}{x^3+x^2} = \dfrac{1}{x^2(x+1)} \sim \dfrac{1}{x^2}$ when $x \simeq 0$

So we guess divergence.

$\dfrac{1}{x^3+x^2} > \dfrac{1}{2x^2}$ if $2x^2 > x^3+x^2$ or $x^2 > x^3$; true if $0 < x < 1$.

$\displaystyle\Longrightarrow \int_0^1 \frac{dx}{x^3+x^2} > \frac{1}{2}\int_0^1 \frac{dx}{x^2}$ which diverges by INT (6)

6B-4 $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ blows up at $x = 1$

$\dfrac{1}{\sqrt{1-x^3}} = \dfrac{1}{\sqrt{(1-x)(1+x+x^2)}} \sim \dfrac{1}{\sqrt{3}\sqrt{1-x}}$ for $x \simeq 1$

So we guess convergence.

$\dfrac{1}{\sqrt{1-x^3}} < \dfrac{1}{\sqrt{1-x}}$ if $x^3 < x$ OK if $0 < x < 1$

$\dfrac{1}{\sqrt{1-x}}$ converges by INT (6), so $\dfrac{1}{\sqrt{1-x^3}}$ also converges by comp.test.

6B-5 $\displaystyle\int_0^\infty \frac{e^{-x}dx}{x}$ is improper at both ends.

At the ∞ end it converges, since

$\dfrac{e^{-x}dx}{x} < e^{-x}$ if $x > 1$ and $\displaystyle\int_0^\infty e^{-x}$converges.

At the 0 end: trouble! $\dfrac{e^{-x}dx}{x} \sim \dfrac{1}{x}$. So we guess divergence.

$\dfrac{e^{-x}dx}{x} > \dfrac{1}{4x}$ on $0 < x < 1 \Longrightarrow \displaystyle\int_0^\infty \frac{e^{-x}dx}{x} > \frac{1}{4}\int_0^\infty \frac{dx}{x}$ divergent.

$\Longrightarrow \displaystyle\int_0^\infty \frac{e^{-x}dx}{x}$ diverges —one end is infinite (the 0 end!)

6B-6 $\displaystyle\int_1^\infty \frac{\ln x dx}{x^2}$

Here $\ln x$ grows so slowly, that we suspect convergence.

$\dfrac{\ln x}{x^2} < \dfrac{x}{x^2}$ is not convergent.

How about $\dfrac{\ln x}{x^2} < \dfrac{1}{x^{3/2}}$? if $x >> 1$. This says $\dfrac{\ln x}{\sqrt{x}} < 1$ if $x >> 1$ and this is true, since

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

$\Longrightarrow \displaystyle\int_1^\infty \frac{\ln x dx}{x^2} < \frac{x}{x^{3/2}}$ converges, by INT (4).

So $\displaystyle\int_1^\infty \frac{\ln x dx}{x^2}$ converges by comp.test.

These have been written out in detail, to review the reasoning. Your own solutions don't have to be so detailed.

6B-7 a) $\displaystyle\int_0^\infty e^{-8x}dx = -(1/8)e^{-8x}\Big|_0^\infty = 1/8$ convergent

b) $\displaystyle\int_1^\infty x^{-n}dx = \frac{x^{-n+1}}{-n+1}\Big|_1^\infty = \frac{1}{n-1}$ convergent ($n > 1$)

c) divergent

d) $\displaystyle\int_0^2 \frac{xdx}{\sqrt{4-x^2}} = -(4-x^2)^{1/2}\Big|_0^2 = 2$ convergent

e) $\displaystyle\int_0^2 \frac{dx}{\sqrt{2-x}} = -2(2-x)^{1/2}\Big|_0^2 = 2\sqrt{2}$ convergent

f) $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^2} = -(\ln x)^{-1}\Big|_e^\infty = 1$ convergent

g) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}} = (3/2)x^{2/3}\Big|_0^1 = \frac{3}{2}$ convergent

h) divergent (at $x = 0$)

i) divergent (at $x = 0$)

j) Convergent because $\ln x$ tends to $-\infty$ more slowly than any power as $x \to 0^+$.

Integrate by parts

$$\int_0^1 \ln x dx = x \ln x - x\Big|_0^1 = -1$$

(Need L'Hospital's rule to check that $x \ln x \to 0$ as $x \to 0^+$.)

k) Convergent because $|e^{-2x} \cos x| < e^{-2x}$. Evaluate by integrating by parts twice (as in E30/4).

$$\int_0^\infty e^{-2x} \cos x dx = \frac{1}{5} e^{-2x} \sin x - \frac{2}{5} e^{-2x} \cos x \Big|_0^\infty = 2/5$$

l) divergent $\left(\int_e^\infty \frac{dx}{x \ln x} = \ln \ln x\Big|_e^\infty = \infty \right)$

m) $\int_0^\infty \frac{dx}{(x+2)^3} = (-1/2)(x+2)^{-2}\Big|_0^\infty = 1/8$ convergent

n) divergent (at $x = 2$)

o) divergent (at $x = 0$)

p) divergent (at $x = \pi/2$)

6B-8 a) $\lim_{x\to\infty} \frac{\int_0^x e^{t^2} dt}{e^{x^2}} = \lim_{x\to\infty} \frac{e^{x^2}}{2xe^{x^2}} = \lim_{x\to\infty} \frac{1}{2x} = 0$ (L'Hospital and FT2)

b) $\lim_{x\to\infty} \frac{\int_0^x e^{t^2} dt}{e^{x^2}/x} = \lim_{x\to\infty} \frac{e^{x^2}}{2x^2e^{x^2} - e^{x^2}/x^2} = \lim_{x\to\infty} \frac{1}{2 - (1/x^2)} = \frac{1}{2}$

c) $\lim_{x\to\infty} \int_0^x e^{-t^2} dt = A$ a finite number > 0 because the integral is convergent. But $e^{x^2} \to \infty$, so the whole limit tends to infinity.

d) $= \lim_{a\to 0^+} \frac{\int_a^1 x^{-1/2} dx}{1/\sqrt{a}} = \lim_{a\to 0^+} \frac{-1/\sqrt{a}}{(-1/2)a^{-3/2}} = \lim_{a\to 0^+} 2a = 0$ (L'Hospital and FT2)

e) $= \lim_{a\to 0^+} \frac{\int_a^1 x^{-3/2} dx}{1/\sqrt{a}} = \lim_{a\to 0^+} \frac{-a^{-3/2}}{(-1/2)a^{-3/2}} = 2$ (L'Hospital and FT2)

(f)

$$\begin{aligned}
\lim_{b\to(\pi/2)^+} (b - \pi/2) \int_0^b \frac{dx}{1 - \sin x} &= \lim_{b\to(\pi/2)^+} \frac{\int_0^b \frac{dx}{1-\sin x}}{1/(b - \pi/2)} \\
&= \lim_{b\to(\pi/2)^+} \frac{1/(1 - \sin b)}{-1/(b - \pi/2)^2} \\
&= \lim_{b\to(\pi/2)^+} \frac{(b - \pi/2)^2}{\sin b - 1} \\
&= \lim_{b\to(\pi/2)^+} \frac{2(b - \pi/2)}{\cos b} \\
&= \lim_{b\to(\pi/2)^+} \frac{2}{-\sin b} = -2
\end{aligned}$$

6C. Infinite Series

6C-1 a) $1+\frac{1}{5}+\frac{1}{25}+\cdots = 1+\frac{1}{5}+\frac{1}{5^2}+\cdots = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$

b) $8+2+\frac{1}{2}+\cdots = 8(1+\frac{1}{4}+\frac{1}{4^2}+\cdots) = 8(\frac{1}{1-\frac{1}{4}}) = \frac{6B}{3}$

c) $\frac{1}{4}+\frac{1}{5}+\cdots = \frac{1}{4}(1+\frac{4}{5}+(\frac{4}{5})^2+\cdot) = \frac{1}{4}(\frac{1}{1-\frac{4}{5}}) = \frac{5}{4}$

d) $0.4444\cdots = 0.4(1+0.1+0.1^2+0.1^3+\cdots) = 0.4(\frac{1}{1-0.1}) = 0.4(\frac{1}{0.9}) = \frac{4}{9}$

e)$0.0602602602\cdots = 0.0602(1+0.001+0.000001+\cdots) = 0.0602(\frac{1}{1-0.001})$

$= \frac{0.0602}{0.999} = \frac{301}{4995}$

6C-2 a) $1+1/2+1/3+1/4+\cdots$

clearly, we have $1 > \int_1^2 \frac{1}{x}dx, \frac{1}{2} > \int_2^3 \frac{1}{x}dx, \cdots$

so we will have $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots > \int_1^2 \frac{1}{x}dx + \int_2^3 \frac{1}{x}dx + \int_3^4 \frac{1}{x}dx + \int_4^5 \frac{1}{x}dx + \cdots =$ $\int_1^\infty \frac{1}{x}dx$, which is divergent, so the infinite series is divergent.

b) $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Case 1: $p \le 1$. $\frac{1}{n^p} > \int_n^{n+1} \frac{dx}{x^p}$

$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^p} > \int_1^\infty \frac{dx}{x^p}$, which is divergent, so the infinite series is divergent.

Case 2: $p > 1$

$\frac{1}{n^p} < \int_{n-1}^{n} \frac{dx}{x^p} \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^p} < 1 + \int_1^\infty \frac{dx}{x^p}$, which is convergent. So the infinite series is convergent.

c) $1/2+1/4+1/6+1/8+\cdots = (1/2)(1+1/2+1/3+1/4+\cdots)$. So from a), the series is divergent.

d) $1+1/3+1/5+1/7+\cdots$

$1 > 1/2, 1/3 > 1/4, 1/5 > 1/6, 1/7 > 1/8, \cdots$

So $1+1/3+1/5+1/7+\cdots > 1/2+1/4+1/6+1/8+\cdots$ which is divergent from c) Thus the series diverges.

(e)

$$\begin{aligned} 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots &= (1-\frac{1}{2})+(\frac{1}{3}-\frac{1}{4})+(\frac{1}{5}-\frac{1}{6})+\cdots \\ &= \frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots \\ &< \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots \\ &< \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots \end{aligned}$$

which is convergent by b). So the infinite series is convergent.

f) $n/n! = 1/(n-1)! < 1/(n-1)(n-2) \simeq 1/n^2$ for $n >> 1$. So convergent by comparison with b).

g) Geometric series with ratio $(\sqrt{5}-1)/2 < 1$, so the series is convergent.

h) Geometric series with ratio $(\sqrt{5}+1)(2\sqrt{5}) < 1$, so the series is convergent.

i) Larger than $\sum 1/n$ for $n \geq 3$, so divergent by part b).

j) $\ln n$ grows more slowly than any power. For instance,

$$\ln n < n^{1/2} \implies \frac{\ln n}{n^2} < n^{-3/2} \quad \text{for } n >> 1$$

The series $\sum n^{-3/2}$ converges by part b), so this series also converges.

k) Converges because $\frac{n+2}{n^4-5} \simeq \frac{1}{n^3}$, and $\sum n^{-3}$ converges by part b).

l) $\frac{(n+2)^{1/3}}{(n^4+5)^{1/3}} \simeq \frac{n^{1/3}}{n^{4/3}} \simeq \frac{1}{n}$. Therefore this series diverges by comparison with $\sum 1/n$.

m) Quadratic approximation implies $\cos(1/n) \approx 1 - 1/2n^2$ and hence

$$\ln(\cos\frac{1}{n}) \simeq -1/2n^2 \quad \text{as } n \to \infty$$

Hence the series converges by comparison with $\sum 1/n^2$ from part b).

n) e^{-n} beats n^2 by a large margin. For example, L'Hospital's rule implies

$$e^{-n/2}n^2 \to 0 \quad \text{as } n \to \infty$$

Therefore for large n, $n^2e^{-n} = n^2e^{-n/2}e^{-n/2} < e^{-n/2}$ and $\sum e^{-n/2}$ is a convergent geometric series. Therefore the original series converges by comparison.

o) Just as in part (n), $e^{-\sqrt{n}}$ beats n^2 by a large margin. L'Hospital's rule implies

$$e^{-m/2}m^4 \to 0 \quad \text{as } m \to \infty$$

Put $m = \sqrt{n}$ to get

$$e^{-\sqrt{n}/2}n^2 \to 0 \quad \text{as } n \to \infty$$

Therefore for large n, $n^2e^{-\sqrt{n}} = n^2e^{-\sqrt{n}/2}e^{-\sqrt{n}/2} < e^{-\sqrt{n}/2}$. Moreover, we also have

$$e^{-\sqrt{n}} < 1/n^2 \qquad n \text{ large}$$

Thus the sum is dominated by $\sum e^{-\sqrt{n}/2} < \sum 1/n^2$ and is convergent by comparison with part b).

6C-3 a)

$$\ln n = \int_1^n \frac{dx}{x} < \text{ Upper sum } = 1 + \frac{1}{2} + \cdots \frac{1}{n-1} < 1 + \frac{1}{2} + \cdots \frac{1}{n}$$

In other words,

$$\ln n < 1 + \frac{1}{2} + \cdots \frac{1}{n}$$

On the other hand,

$$\ln n = \int_1^n \frac{dx}{x} > \text{ Lower sum } = \frac{1}{2} + \cdots \frac{1}{n}$$

Adding 1 to both sides,

$$1 + \ln n > 1 + \frac{1}{2} + \cdots \frac{1}{n}$$

b) Need at least $\ln n = 999$

$$\text{Time} > 10^{-10}e^{999} \approx 7 \times 10^{423} \text{ seconds}$$

This is far, far longer than the estimated time from the “big bang.”