Unit 3. Integration

3A. Differentials, indefinite integration

3A-1 a) $7x^6 dx$. ($d(\sin 1) = 0$ because $\sin 1$ is a constant.)

b) $(1/2)x^{-1/2}dx$

c) $(10x^9 - 8)dx$

d) $(3e^{3x}\sin x + e^{3x}\cos x)dx$

e) $(1/2\sqrt{x})dx + (1/2\sqrt{y})dy = 0$ implies

$$dy = -\frac{1/2\sqrt{x}dx}{1/2\sqrt{y}} = -\frac{\sqrt{y}}{\sqrt{x}}dx = -\frac{1-\sqrt{x}}{\sqrt{x}}dx = \left(1 - \frac{1}{\sqrt{x}}\right)dx$$

3A-2 a) $(2/5)x^5 + x^3 + x^2/2 + 8x + c$

b) $(2/3)x^{3/2} + 2x^{1/2} + c$

c) Method 1 (slow way) Substitute: $u = 8 + 9x$, $du = 9dx$. Therefore

$$\int \sqrt{8+9x}dx = \int u^{1/2}(1/9)du = (1/9)(2/3)u^{3/2} + c = (2/27)(8+9x)^{3/2} + c$$

Method 2 (guess and check): It's often faster to guess the form of the antiderivative and work out the constant factor afterwards:

$$\text{Guess } (8+9x)^{3/2}; \quad \frac{d}{dx}(8+9x)^{3/2} = (3/2)(9)(8+9x)^{1/2} = \frac{27}{2}(8+9x)^{1/2}.$$

So multiply the guess by $\frac{2}{27}$ to make the derivative come out right; the answer is then

$$\frac{2}{27}(8+9x)^{3/2} + c$$

d) Method 1 (slow way) Use the substitution: $u = 1 - 12x^4$, $du = -48x^3dx$.

$$\int x^3(1-12x^4)^{1/8}dx = \int u^{1/8}(-1/48)du = -\frac{1}{48}(8/9)u^{9/8} + c = -\frac{1}{54}(1-12x^4)^{9/8} + c$$

Method 2 (guess and check): guess $(1-12x^4)^{9/8}$;

$$\frac{d}{dx}(1-12x^4)^{9/8} = \frac{9}{8}(-48x^3)(1-12x^4)^{1/8} = -54(1-12x^4)^{1/8}.$$

So multiply the guess by $-\frac{1}{54}$ to make the derivative come out right, getting the previous answer.

e) Method 1 (slow way): Use substitution: $u = 8 - 2x^2$, $du = -4xdx$.

$$\int \frac{x}{\sqrt{8-2x^2}}dx = \int u^{1/2}(-1/4)du = -\frac{1}{4}\frac{2}{3}u^{3/2} + c = -\frac{1}{6}(8-2x^2)^{3/2} + c$$

Method 2 (guess and check): guess $(8-2x^2)^{3/2}$; differentiating it:

$$\frac{d}{dx}(8-2x^2)^{3/2} = \frac{3}{2}(-4x^2)(8-2x^2)^{1/2} = -6(8-2x^2)^{1/2};$$

so multiply the guess by $-\dfrac{1}{6}$ to make the derivative come out right.

The next four questions you should try to do (by Method 2) in your head. Write down the correct form of the solution and correct the factor in front.

f) $(1/7)e^{7x} + c$

g) $(7/5)e^{x^5} + c$

h) $2e^{\sqrt{x}} + c$

i) $(1/3)\ln(3x+2) + c$. For comparison, let's see how much slower substitution is:

$$u = 3x+2, \quad du = 3dx, \quad \text{so}$$

$$\int \frac{dx}{3x+2} = \int \frac{(1/3)du}{u} = (1/3)\ln u + c = (1/3)\ln(3x+2) + c$$

j)

$$\int \frac{x+5}{x}dx = \int \left(1 + \frac{5}{x}\right) dx = x + 5\ln x + c$$

k)

$$\int \frac{x}{x+5}dx = \int \left(1 - \frac{5}{x+5}\right) dx = x - 5\ln(x+5) + c$$

In Unit 5 this sort of algebraic trick will be explained in detail as part of a general method. What underlies the algebra in both (j) and (k) is the algorithm of long division for polynomials.

l) $u = \ln x$, $du = dx/x$, so

$$\int \frac{\ln x}{x}dx = \int udu = (1/2)u^2 + c = (1/2)(\ln x)^2 + c$$

m) $u = \ln x$, $du = dx/x$.

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln u + c = \ln(\ln x) + c$$

3A-3 a) $-(1/5)\cos(5x) + c$

b) $(1/2)\sin^2 x + c$, coming from the substitution $u = \sin x$ or $-(1/2)\cos^2 x + c$, coming from the substitution $u = \cos x$. The two functions $(1/2)\sin^2 x$ and $-(1/2)\cos^2 x$ are not the same. Nevertheless the two answers given are the same. Why? (See 1J-1(m).)

c) $-(1/3)\cos^3 x + c$

d) $-(1/2)(\sin x)^{-2} + c = -(1/2)\csc^2 x + c$

e) $5\tan(x/5)+c$

f) $(1/7)\tan^7 x + c$.

g) $u=\sec x,\ du=\sec x\tan x dx$,

$$\int \sec^9 x\tan x dx \int (\sec x)^8 \sec x\tan x dx = (1/9)\sec^9 x + c$$

3B. Definite Integrals

3B-1 a) $1+4+9+16=30$ b) $2+4+8+16+32+64=126$

c) $-1+4-9+16-25=-15$ d) $1+1/2+1/3+1/4=25/12$

3B-2 a) $\sum_{n=1}^{6}(-1)^{n+1}(2n+1)$ b) $\sum_{k=1}^{n} 1/k^2$ c) $\sum_{k=1}^{n}\sin(kx/n)$

3B-3 a) upper sum = right sum = $(1/4)[(1/4)^3+(2/4)^3+(3/4)^3+(4/4)^3]=15/128$

lower sum = left sum = $(1/4)[0^3+(1/4)^3+(2/4)^3+(3/4)^3]=7/128$

b) left sum $=(-1)^2+0^2+1^2+2^2=6$; right sum $=0^2+1^2+2^2+3^2=14$;

upper sum $=(-1)^2+1^2+2^2+3^2=15$; lower sum $=0^2+0^2+1^2+2^2=5$.

c) left sum $=(\pi/2)[\sin 0+\sin(\pi/2)+\sin(\pi)+\sin(3\pi/2)]=(\pi/2)[0+1+0-1]=0$;

right sum $=(\pi/2)[\sin(\pi/2)+\sin(\pi)+\sin(3\pi/2)+\sin(2\pi)]=(\pi/2)[1+0-1+0]=0$;

upper sum $=(\pi/2)[\sin(\pi/2)+\sin(\pi/2)+\sin(\pi)+\sin(2\pi)]=(\pi/2)[1+1+0+0]=\pi$;

lower sum $=(\pi/2)[\sin(0)+\sin(\pi)+\sin(3\pi/2)+\sin(3\pi/2)]=(\pi/2)[0+0-1-1]=-\pi$.

3B-4 Both x^2 and x^3 are increasing functions on $0\le x\le b$, so the upper sum is the right sum and the lower sum is the left sum. The difference between the right and left Riemann sums is

$$(b/n)[f(x_1+\cdots+f(x_n)]-(b/n)[f(x_0+\cdots+f(x_{n-1})]=(b/n)[f(x_n)-f(x_0)]$$

In both cases $x_n=b$ and $x_0=0$, so the formula is

$$(b/n)(f(b)-f(0))$$

a) $(b/n)(b^2-0)=b^3/n$. Yes, this tends to zero as $n\to\infty$.

b) $(b/n)(b^3-0)=b^4/n$. Yes, this tends to zero as $n\to\infty$.

3B-5 The expression is the right Riemann sum for the integral

$$\int_0^1 \sin(bx)dx = -(1/b)\cos(bx)|_0^1 = (1-\cos b)/b$$

so this is the limit.

3C. Fundamental theorem of calculus

3C-1

$$\int_3^6 (x-2)^{-1/2}dx = 2(x-2)^{1/2}\Big|_3^6 = 2[(4)^{1/2} - 1^{1/2}] = 2$$

3C-2 a) $(2/3)(1/3)(3x+5)^{3/2}\Big|_0^2 = (2/9)(11^{3/2} - 5^{3/2})$

b) If $n \neq -1$, then

$$(1/(n+1))(1/3)(3x+5)^{n+1}\Big|_0^2 = (1/3(n+1))((11^{n+1} - 5^{n+1})$$

If $n = -1$, then the answer is $(1/3)\ln(11/5)$.

c) $(1/2)(\cos x)^{-2}\Big|_{3\pi/4}^{\pi} = (1/2)[(-1)^{-2} - (-1/\sqrt{2})^{-2}] = -1/2$

3C-3 a) $(1/2)\ln(x^2+1)\Big|_1^2 = (1/2)[\ln 5 - \ln 2] = (1/2)\ln(5/2)$

b) $(1/2)\ln(x^2+b^2)\Big|_b^{2b} = (1/2)[\ln(5b^2) - \ln(2b^2)] = (1/2)\ln(5/2)$

3C-4 As $b \to \infty$,

$$\int_1^b x^{-10}dx = -(1/9)x^{-9}\Big|_1^b = -(1/9)(b^{-9} - 1) \to -(1/9)(0-1) = 1/9.$$

This integral is the area of the infinite region between the curve $y = x^{-10}$ and the x-axis for $x > 0$.

3C-5 a) $\displaystyle\int_0^{\pi} \sin x dx = -\cos x\Big|_0^{\pi} = -(\cos\pi - \cos 0) = 2$

b) $\displaystyle\int_0^{\pi/a} \sin(ax)dx = -(1/a)\cos(ax)\Big|_0^{\pi/a} = -(1/a)(\cos\pi - \cos 0) = 2/a$

3C-6 a) $x^2 - 4 = 0$ implies $x = \pm 2$. So the area is

$$\int_{-2}^{2}(x^2-4)dx = 2\int_0^2 (x^2-4)dx = \frac{x^3}{3} - 4x\Big|_0^2 = \frac{8}{3} - 4\cdot 2 = -16/3$$

(We changed to the interval $(0,2)$ and doubled the integral because x^2-4 is even.) Notice that the integral gave the wrong answer! It's negative. This is because the graph $y = x^2-4$ is concave up and is below the x-axis in the interval $-2 < x < 2$. So the correct answer is $16/3$.

b) Following part (a), $x^2 - a = 0$ implies $x = \pm\sqrt{a}$. The area is

$$\int_{-\sqrt{a}}^{\sqrt{a}}(a-x^2)dx = 2\int_0^{\sqrt{a}}(a-x^2)dx = 2ax - \frac{x^3}{3}\Big|_0^{\sqrt{a}} = 2\big(a^{3/2} - \frac{a^{3/2}}{3}\big) = \frac{4}{3}a^{3/2}$$

3D. Second fundamental theorem

3D-1 Differentiate both sides;

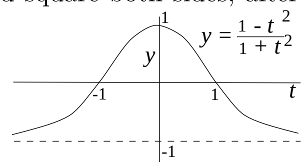
left side $L(x)$: $L'(x) = \dfrac{d}{dx}\displaystyle\int_0^x \frac{dt}{\sqrt{a^2+x^2}} = \frac{1}{\sqrt{a^2+x^2}}$, by FT2;

right side $R(x)$: $R'(x) = \dfrac{d}{dx}(\ln(x+\sqrt{a^2+x^2}) - \ln a) = \dfrac{1+\frac{x}{\sqrt{a^2+x^2}}}{x+\sqrt{a^2+x^2}} = \dfrac{1}{\sqrt{a^2+x^2}}$

Since $L'(x) = R'(x)$, we have $L(x) = R(x) + C$ for some constant $C = L(x) - R(x)$. The constant C may be evaluated by assigning a value to x; the most convenient choice is $x = 0$, which gives

$$L(0) = \int_0^0 = 0;\quad R(0) = \ln(0+\sqrt{0+a^2}) - \ln a = 0;\quad \text{therefore } C = 0 \text{ and } L(x) = R(x).$$

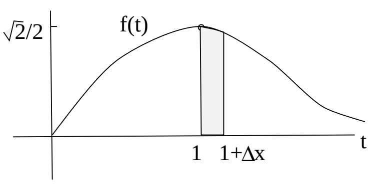
b) Put $x = c$; the equation becomes $0 = \ln(c+\sqrt{c^2+a^2})$; solve this for c by first exponentiating both sides: $1 = c+\sqrt{c^2+a^2}$; then subtract c and square both sides; after some algebra one gets $c = \frac{1}{2}(1-a^2)$.



3D-3 Sketch $y = \dfrac{1-t^2}{1+t^2}$ first, as shown at the right.

3D-4 a) $\displaystyle\int_0^x \sin(t^3)dt$, by the FT2. b) $\displaystyle\int_0^x \sin(t^3)dt + 2$ c) $\displaystyle\int_1^x \sin(t^3)dt - 1$

3D-5 This problem reviews the idea of the proof of the FT2.



a) $f(t) = \dfrac{t}{\sqrt{1+t^4}}$

$$\frac{1}{\Delta x}\int_1^{1+\Delta x} f(t)dt = \frac{\text{shaded area}}{\text{width}} \approx \text{height} \quad .$$

$$\lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_1^{1+\Delta x} f(t)dt = \lim_{\Delta x\to 0}\frac{\text{shaded area}}{\text{width}} = \text{height} = f(1) = \frac{1}{\sqrt{2}}.$$

b) By definition of derivative,

$$F'(1) = \lim_{\Delta x\to 0}\frac{F(1+\Delta x)-F(1)}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_1^{1+\Delta x} f(t)dt;$$

by FT2, $F'(1) = f(1) = \dfrac{1}{\sqrt{2}}$.

3D-6 a) If $F_1(x) = \displaystyle\int_{a_1}^x dt$ and $F_2(x) = \displaystyle\int_{a_2}^x dt$, then $F_1(x) = x - a_1$ and $F_2(x) = x - a_2$. Thus $F_1(x) - F_2(x) = a_2 - a_1$, a constant.

b) By the FT2, $F_1'(x) = f(x)$ and $F_2'(x) = f(x)$; therefore $F_1 = F_2 + C$, for some constant C.

3D-7 a) Using the FT2 and the chain rule, as in the Notes,

$$\frac{d}{dx}\int_0^{x^2}\sqrt{u}\sin u\,du = \sqrt{x^2}\sin(x^2)\cdot\frac{d(x^2)}{dx} = 2x^2\sin(x^2)$$

b) $= \dfrac{1}{\sqrt{1-\sin^2 x}}\cdot\cos x = 1.$ (So $\displaystyle\int_0^{\sin x}\frac{dt}{1-t^2} = x$)

c) $\displaystyle\frac{d}{dx}\int_x^{x^2}\tan u\,du = \tan(x^2)\cdot 2x - \tan x$

3D-8 a) Differentiate both sides using FT2, and substitute $x = \pi/2$: $f(\pi/2) = 4$.

b) Substitute $x = 2u$ and follow the method of part (a); put $u = \pi$, get finally $f(\pi/2) = 4 - 4\pi$.

3E. Change of Variables; Estimating Integrals

3E-1 $L(\frac{1}{a}) = \displaystyle\int_1^{1/a}\frac{dt}{t}$. Put $t = \frac{1}{u}$, $dt = -\frac{1}{u^2}du$. Then

$$\frac{dt}{t} = -\frac{u}{u^2}du \implies L(\frac{1}{a}) = \int_1^{1/a}\frac{dt}{t} = -\int_1^a\frac{du}{u} = -L(a)$$

3E-2 a) We want $-t^2 = -u^2/2$, so $u = t\sqrt{2}$, $du = \sqrt{2}dt$.

$$\frac{1}{\sqrt{2\pi}}\int_0^x e^{-u^2/2}du = \frac{1}{\sqrt{2\pi}}\int_0^{x/\sqrt{2}} e^{-t^2}\sqrt{2}dt = \frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{2}} e^{-t^2}dt$$

$$\implies E(x) = \frac{1}{\sqrt{\pi}}F(x/\sqrt{2}) \quad \text{and} \quad \lim_{x\to\infty}E(x) = \frac{1}{\pi}\cdot\frac{\sqrt{\pi}}{2} = \frac{1}{2}$$

b) The integrand is even, so

$$\frac{1}{\sqrt{2\pi}}\int_{-N}^N e^{-u^2/2}du = \frac{2}{\sqrt{2\pi}}\int_0^N e^{-u^2/2}du = 2E(N) \longrightarrow 1 \quad \text{as } N\to\infty$$

$$\lim_{x\to-\infty}E(x) = -1/2 \quad \text{because } E(x) \text{ is odd.}$$

$$\frac{1}{\sqrt{2\pi}}\int_a^b e^{-u^2/2}du = E(b) - E(a) \quad \text{by FT1 or by “interval addition” Notes PI (3).}$$

Commentary: The answer is consistent with the limit,

$$\frac{1}{\sqrt{2\pi}}\int_{-N}^N e^{-u^2/2}du = E(N) - E(-N) = 2E(N) \longrightarrow 1 \text{ as } N\to\infty$$

3E-3 a) Using $u = \ln x,\ du = \dfrac{dx}{x}$, $\displaystyle\int_1^e \frac{\sqrt{\ln x}}{x}dx = \int_0^1 \sqrt{u}du = \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{3}$.

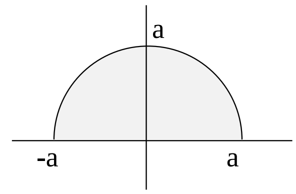
b) Using $u = \cos x,\ du = -\sin x$,

$$\int_0^\pi \frac{\sin x}{(2+\cos x)^3}dx = \int_1^{-1} \frac{-du}{(2+u)^3} = \frac{1}{2(2+u)^2}\Big|_1^{-1} = \frac{1}{2}\left(\frac{1}{1^2} - \frac{1}{3^2}\right) = \frac{4}{9}..$$

c) Using $x = \sin u,\ dx = \cos u du$, $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_0^{\frac{\pi}{2}} \frac{\cos u}{\cos u}du = u\Big|_0^{\pi/2} = \frac{\pi}{2}$.

3E-4 Substitute $x = t/a$; then $x = \pm 1 \ \Rightarrow\ t = \pm a$. We then have

$$\frac{\pi}{2} = \int_{-1}^1 \sqrt{1-x^2}dx = \int_{-a}^a \sqrt{1 - \frac{t^2}{a^2}}\frac{dt}{a} = \frac{1}{a^2}\int_{-a}^a \sqrt{a^2-t^2}dt.$$



Multiplying by a^2 gives the value $\pi a^2/2$ for the integral, which checks, since the integral represents the area of the semicircle.

3E-5 One can use informal reasoning based on areas (as in Ex. 5, Notes FT), but it is better to use change of variable.

a) Goal: $F(-x) = -F(x)$. Let $t = -u$, $dt = -du$, then

$$F(-x) = \int_0^{-x} f(t)dt = \int_0^x f(-u)(-du)$$

Since f is even ($f(-u) = f(u)$), $F(-x) = -\displaystyle\int_0^x f(u)du = -F(x)$.

b) Goal: $F(-x) = F(x)$. Let $t = -u$, $dt = -du$, then

$$F(-x) = \int_0^{-x} f(t)dt = \int_0^x f(-u)(-du)$$

Since f is odd (($f(-u) = -f(u)$), $F(-x) = \displaystyle\int_0^x f(u)du = F(x)$.

3E-6 a) $x^3 < x$ on (0,1) $\Rightarrow\ \dfrac{1}{1+x^3} > \dfrac{1}{1+x}$ on (0,1); therefore

$$\int_0^1 \frac{dx}{1+x^3} > \int_0^1 \frac{dx}{1+x} = \ln(1+x)\Big|_0^1 = \ln 2 = .69$$

b) $0 < \sin x < 1$ on $(0,\pi)$ $\Rightarrow\ \sin^2 x < \sin x$ on $(0,\pi)$; therefore

$$\int_0^\pi \sin^2 x dx < \int_0^\pi \sin x dx = -\cos x\Big|_0^\pi = -(-1-1) = 2.$$

c) $\displaystyle\int_{10}^{20} \sqrt{x^2+1}dx > \int_{10}^{20}\sqrt{x^2}dx = \frac{x^2}{2}\Big|_{10}^{20} = \frac{1}{2}(400-100) = 150$

3E-7 $\displaystyle\left|\int_1^N \frac{\sin x}{x^2}dx\right| \le \int_1^N \frac{|\sin x|}{x^2}dx \le \int_1^N \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^N = -\frac{1}{N} + 1 < 1.$

3F. Differential Equations: Separation of Variables. Applications

3F-1 a) $y = (1/10)(2x+5)^5 + c$

b) $(y+1)dy = dx \implies \int (y+1)dy = \int dx \implies (1/2)(y+1)^2 = x + c$. You can leave this in implicit form or solve for y: $y = -1 \pm \sqrt{2x+a}$ for any constant a ($a = 2c$)

c) $y^{1/2}dy = 3dx \implies (2/3)y^{3/2} = 3x + c \implies y = (9x/2 + a)^{2/3}$, with $a = (3/2)c$.

d) $y^{-2}dy = xdx \implies -y^{-1} = x^2/2 + c \implies y = -1/(x^2/2 + c)$

3F-2 a) Answer: $3e^{16}$.

$$y^{-1}dy = 4xdx \implies \ln y = 2x^2 + c$$

$$y(1) = 3 \implies \ln 3 = 2 + c \implies c = \ln 3 - 2.$$

Therefore

$$\ln y = 2x^2 + (\ln 3 - 2)$$

At $x = 3$, $y = e^{18+\ln 3-2} = 3e^{16}$

b) Answer: $y = 11/2 + 3\sqrt{2}$.

$$(y+1)^{-1/2}dy = dx \implies 2(y+1)^{1/2} = x + c$$

$$y(0) = 1 \implies 2(1+1)^{1/2} = c \implies c = 2\sqrt{2}$$

At $x = 3$,

$$2(y+1)^{1/2} = 3 + 2\sqrt{2} \implies y + 1 = (3/2 + \sqrt{2})^2 = 13/2 + 3\sqrt{2}$$

Thus, $y = 11/2 + 3\sqrt{2}$.

c) Answer: $y = \sqrt{550/3}$

$$ydy = x^2dx \implies y^2/2 = (1/3)x^3 + c$$

$$y(0) = 10 \implies c = 10^2/2 = 50$$

Therefore, at $x = 5$,

$$y^2/2 = (1/3)5^3 + 50 \implies y = \sqrt{550/3}$$

d) Answer: $y = (2/3)(e^{24} - 1)$

$$(3y+2)^{-1}dy = dx \implies (1/3)\ln(3y+2) = x + c$$

$$y(0) = 0 \implies (1/3)\ln 2 = c$$

Therefore, at $x = 8$,

$$(1/3)\ln(3y+2) = 8 + (1/3)\ln 2 \implies \ln(3y+2) = 24 + \ln 2 \implies (3y+2) = 2e^{24}$$

Therefore, $y = (2e^{24} - 2)/3$

e) Answer: $y = -\ln 4$ at $x = 0$. Defined for $-\infty < x < 4$.

$$e^{-y}dy = dx \implies -e^{-y} = x + c$$

$$y(3) = 0 \implies -e^0 = 3 + c \implies c = -4$$

Therefore,

$$y = -\ln(4 - x), \quad y(0) = -\ln 4$$

The solution y is defined only if $x < 4$.

3F-3 a) Answers: $y(1/2) = 2$, $y(-1) = 1/2$, $y(1)$ is undefined.

$$y^{-2}dy = dx \implies -y^{-1} = x + c$$

$$y(0) = 1 \implies -1 = 0 + c \implies c = -1$$

Therefore, $-1/y = x - 1$ and

$$y = \frac{1}{1 - x}$$

The values are $y(1/2) = 2$, $y(-1) = -1/2$ and y is undefined at $x = 1$.

b) Although the formula for y makes sense at $x = 3/2$, $(y(3/2) = 1/(1 - 3/2) = -2)$, it is not consistent with the rate of change interpretation of the differential equation. The function is defined, continuous and differentiable for $-\infty < x < 1$. But at $x = 1$, y and dy/dx are undefined. Since $y = 1/(1 - x)$ is the only solution to the differential equation in the interval $(0, 1)$ that satisfies the initial condition $y(0) = 1$, it is impossible to define a function that has the initial condition $y(0) = 1$ and also satisfies the differential equation in any longer interval containing $x = 1$.

To ask what happens to y after $x = 1$, say at $x = 3/2$, is something like asking what happened to a rocket ship after it fell into a black hole. There is no obvious reason why one has to choose the formula $y = 1/(1 - x)$ after the "explosion." For example, one could define $y = 1/(2 - x)$ for $1 \le x < 2$. In fact, any formula $y = 1/(c - x)$ for $c \ge 1$ satisfies the differential equation at every point $x > 1$.

3F-4 a) If the surrounding air is cooler ($T_e - T < 0$), then the object will cool, so $dT/dt < 0$. Thus $k > 0$.

b) Separate variables and integrate.

$$(T - T_e)^{-1}dT = -kdt \implies \ln|T - T_e| = -kt + c$$

Exponentiating,

$$T - T_e = \pm e^c e^{-kt} = Ae^{-kt}$$

The initial condition $T(0) = T_0$ implies $A = T_0 - T_e$. Thus

$$T = T_e + (T_0 - T_e)e^{-kt}$$

c) Since $k > 0$, $e^{-kt} \to 0$ as $t \to \infty$. Therefore,

$$T = T_e + (T_0 - T_e)e^{-kt} \longrightarrow T_e \text{ as } t \to \infty$$

d)
$$T - T_e = (T_0 - T_e)e^{-kt}$$

The data are $T_0 = 680$, $T_e = 40$ and $T(8) = 200$. Therefore,

$$200 - 40 = (680 - 40)e^{-8k} \implies e^{-8k} = 160/640 = 1/4 \implies -8k = -\ln 4.$$

The number of hours t that it takes to cool to $50°$ satisfies the equation

$$50 - 40 = (640)e^{-kt} \implies e^{-kt} = 1/64 \implies -kt = -3\ln 4.$$

To solve the two equations on the right above simultaneously for t, it is easiest just to divide the bottom equation by the top equation, which gives

$$\frac{t}{8} = 3, \quad t = 24.$$

e)
$$T - T_e = (T_0 - T_e)e^{-kt}$$

The data at $t = 1$ and $t = 2$ are

$$800 - T_e = (1000 - T_e)e^{-k} \quad \text{and} \quad 700 - T_e = (1000 - T_e)e^{-2k}$$

Eliminating e^{-k} from these two equations gives

$$\begin{aligned}
\frac{700 - T_e}{1000 - T_e} &= \left(\frac{800 - T_e}{1000 - T_e}\right)^2 \\
(800 - T_e)^2 &= (1000 - T_e)(700 - T_e) \\
800^2 - 1600T_e + T_e^2 &= (1000)(700) - 1700T_e + T_e^2 \\
100T_e &= (1000)(700) - 800^2 \\
T_e &= 7000 - 6400 = 600
\end{aligned}$$

f) To confirm the differential equation:

$$y'(t) = T'(t - t_0) = k(T_e - T(t - t_0)) = k(T_e - y(t))$$

The formula for y is

$$y(t) = T(t - t_0) = T_e + (T_0 - T_e)e^{-k(t-t_0)} = a + (y(t_0) - a)e^{-c(t-t_0)}$$

with $k = c$, $T_e = a$ and $T_0 = T(0) = y(t_0)$.

3F-6 $y = \cos^3 u - 3\cos u$, $x = \sin^4 u$

$dy = (3\cos^2 u \cdot (-\sin u) + 3\sin u)du$, $dx = 4\sin^3 u \cos u du$

$\dfrac{dy}{dx} = \dfrac{3\sin u(1 - \cos^2 u)}{4\sin^3 u \cos u} = \dfrac{3}{4\cos u}$

3F-7 a) $y' = -xy$; $y(0) = 1$

$$\frac{dy}{y} = -xdx \implies \ln y = -\frac{1}{2}x^2 + c$$

To find c, put $x = 0$, $y = 1$: $\ln 1 = 0 + c \implies c = 0$.

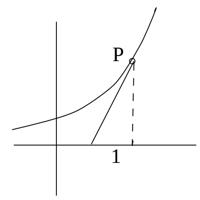
$$\implies \ln y = -\frac{1}{2}x^2 \implies y = e^{-x^2/2}$$

b) $\cos x \sin y dy = \sin x dx$; $y(0) = 0$

$$\sin y dy = \frac{\sin x}{\cos x} dx \implies -\cos y = -\ln(\cos x) + c$$

Find c: put $x = 0$, $y = 0$: $-\cos 0 = -\ln(\cos 0) + c \implies c = -1$

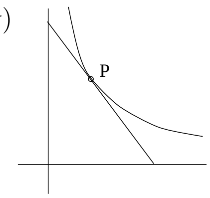
$$\implies \cos y = \ln(\cos x) + 1$$



3F-8 a) From the triangle, $y' = \text{slope tangent} = \dfrac{y}{1}$

$$\implies \frac{dy}{y} = dx \implies \ln y = x + c_1 \implies y = e^{x+c_1} = Ae^x \ \ (A = e^{c_1})$$

b) If P bisects tangent, then P_0 bisects OQ (by euclidean geometry)



So $P_0Q = x$ (since $OP_0 = x$).

$$\text{Slope tangent} = y' = \frac{-y}{x} \implies \frac{dy}{y} = -\frac{dx}{x}$$

$$\implies \ln y = -\ln x + c_1$$

Exponentiate: $y = \dfrac{1}{x} \cdot e^{c_1} = \dfrac{c}{x}, c > 0$

Ans: The hyperbolas $y = \dfrac{c}{x}, c > 0$

3G. Numerical Integration

3G-1 Left Riemann sum: $(\Delta x)(y_0 + y_1 + y_2 + y_3)$

Trapezoidal rule: $(\Delta x)((1/2)y_0 + y_1 + y_2 + y_3 + (1/2)y_4)$

Simpson's rule: $(\Delta x/3)(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$

a) $\Delta x = 1/4$ and

$$y_0 = 0,\ y_1 = 1/2,\ y_2 = 1/\sqrt{2},\ y_3 = \sqrt{3}/2,\ y_4 = 1.$$

Left Riemann sum: $(1/4)(0 + 1/2 + 1/\sqrt{2} + \sqrt{3}/2) \approx .518$

Trapezoidal rule: $(1/4)((1/2) \cdot 0 + 1/2 + 1/\sqrt{2} + \sqrt{3}/2 + (1/2)1) \approx .643$

Simpson's rule: $(1/12)(1 \cdot 0 + 4(1/2) + 2(1/\sqrt{2}) + 4(\sqrt{3}/2) + 1) \approx .657$

as compared to the exact answer $.6666\ldots$

b) $\Delta x = \pi/4$

$$y_0 = 0,\ y_1 = 1/\sqrt{2},\ y_2 = 1,\ y_3 = 1/\sqrt{2},\ y_4 = 0.$$

Left Riemann sum: $(\pi/4)(0 + 1/\sqrt{2} + 1 + 1/\sqrt{2}) \approx 1.896$

Trapezoidal rule: $(\pi/4)((1/2)\cdot 0 + 1/\sqrt{2} + 1 + 1/\sqrt{2} + (1/2)\cdot 0) \approx 1.896$ (same as Riemann sum)

Simpson's rule: $(\pi/12)(1\cdot 0 + 4(1/\sqrt{2}) + 2(1) + 4(1/\sqrt{2}) + 1\cdot 0) \approx 2.005$

as compared to the exact answer 2

c) $\Delta x = 1/4$

$$y_0 = 1,\ y_1 = 16/17,\ y_2 = 4/5,\ y_3 = 16/25,\ y_4 = 1/2.$$

Left Riemann sum: $(1/4)(1 + 16/17 + 4/5 + 16/25) \approx .845$

Trapezoidal rule: $(1/4)((1/2)\cdot 1 + 16/17 + 4/5 + 16/25 + (1/2)(1/2)) \approx .8128$

Simpson's rule: $(1/12)(1\cdot 1 + 4(16/17) + 2(4/5) + 4(16/25) + 1(1/2)) \approx .785392$

as compared to the exact answer $\pi/4 \approx .785398$

(Multiplying the Simpson's rule answer by 4 gives a passable approximation to π, of 3.14157, accurate to about 2×10^{-5}.)

d) $\Delta x = 1/4$

$$y_0 = 1,\ y_1 = 4/5,\ y_2 = 2/3,\ y_3 = 4/7,\ y_4 = 1/2.$$

Left Riemann sum: $(1/4)(1 + 4/5 + 2/3 + 4/7) \approx .76$

Trapezoidal rule: $(1/4)((1/2)\cdot 1 + 4/5 + 2/3 + 4/7(1/2)(1/2)) \approx .697$

Simpson's rule: $(1/12)(1\cdot 1 + 4(4/5) + 2(2/3) + 4(4/7) + 1(1/2)) \approx .69325$

Compared with the exact answer $\ln 2 \approx .69315$, Simpson's rule is accurate to about 10^{-4}.

3G-2 We have $\displaystyle\int_0^b x^3dx = \frac{b^4}{4}$. Using Simpson's rule with two subintervals, $\Delta x = b/2$, so that we get the same answer as above:

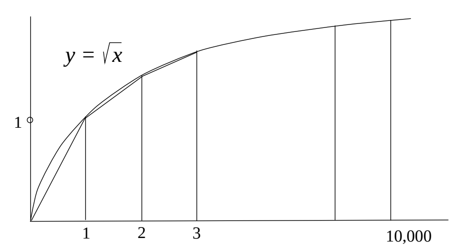
$$S(x^3) = \frac{b}{6}(0 + 4(b/2)^3 + b^3) = \frac{b}{6}\left(\frac{3}{2}b^3\right) = \frac{b^4}{4}.$$

Remark. The fact that Simpson's rule is exact on cubic polynomials is very significant to its effectiveness as a numerical approximation. It implies that the approximation converges at a rate proportional to the the fourth derivative of the function times $(\Delta x)^4$, which is fast enough for many practical purposes.

3G-3 The sum

$$S = \sqrt{1} + \sqrt{2} + ... + \sqrt{10,000}$$

is related to the trapezoidal estimate of $\displaystyle\int_0^{10^4} \sqrt{x}dx$:



(1) $$\int_0^{10^4} \sqrt{x}dx \approx \frac{1}{2}\sqrt{0} + \sqrt{1} + ... + \frac{1}{2}\sqrt{10^4} = S - \frac{1}{2}\sqrt{10^4}$$

But

$$\int_0^{10^4} \sqrt{x}dx = \frac{2}{3}x^{3/2}\Big|_0^{10^4} = \frac{2}{3} \cdot 10^6$$

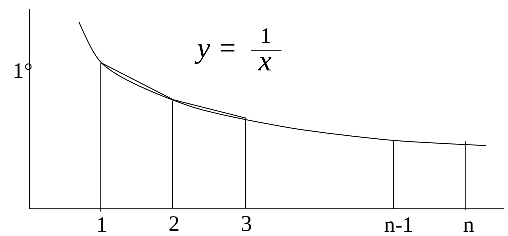
From (1),

(2) $$\frac{2}{3} \cdot 10^6 \approx S - 50$$

Hence

(3) $$S \approx 666,717$$

In (1), we have $>$, as in the picture. Hence in (2), we have $>$, so in (3), we have $<$, Too high.



3G-4 As in Problem 3 above, let

$$S = \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n}$$

Then by trapezoidal rule,

$$\int_1^n \frac{dx}{x} \approx \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2} \cdot \frac{1}{n} = S - \frac{1}{2} - \frac{1}{2n}$$

Since $\displaystyle\int_1^n \frac{dx}{x} = \ln n$, we have $S \approx \ln n + \frac{1}{2} + \frac{1}{2n}$. (Estimate is too low.)

3G-5 Referring to the two pictures above, one can see that if $f(x)$ is concave down on $[a, b]$, the trapezoidal rule gives too low an estimate; if $f(x)$ is concave up, the trapezoidal rule gives too high an estimate..