2. Applications of Differentiation

2A. Approximation

2A-1 $\frac{d}{dx}\sqrt{a+bx} = \frac{b}{2\sqrt{a+bx}} \quad \Rightarrow \ f(x) \approx \sqrt{a} + \frac{b}{2\sqrt{a}}x$ by formula.

By algebra: $\sqrt{a+bx} = \sqrt{a}\sqrt{1+\frac{bx}{a}} \approx \sqrt{a}(1+\frac{bx}{2a})$, same as above.

2A-2 $D(\frac{1}{a+bx}) = \frac{-b}{(a+bx)^2} \ \Rightarrow \ f(x) \approx \frac{1}{a} - \frac{b}{a^2}x$; OR: $\frac{1}{a+bx} = \frac{1/a}{1+b/ax} \ \approx \ \frac{1}{a}(1-\frac{b}{a}x)$.

2A-3 $D(\frac{(1+x)^{3/2}}{1+2x}) = \frac{(1+2x)\cdot\frac{3}{2}\cdot(1+x)^{3/2} - (1+x)^{3/2}\cdot 2}{(1+2x)^2} \Rightarrow f'(0) = -\frac{1}{2}$

$\Rightarrow \ f(x) \approx 1 - \frac{1}{2}x$; OR, by algebra, $\frac{(1+x)^{3/2}}{1+2x} \approx (1+\frac{3}{2}x)(1-2x) \approx 1 - \frac{1}{2}x.$

2A-4 Put $\frac{h}{R} = \epsilon$; then $w = \frac{g}{(1+\epsilon)^2} \ \approx \ g(1-\epsilon)^2 \ \approx \ g(1-2\epsilon) = g(1-\frac{2h}{R}).$

2A-5 A reasonable assumption is that w is propotional to volume v, which is in turn proportional to the *cube* of a linear dimension, i.e., a given person remains similar to him/herself, for *small* weight changes.) Thus $w = Ch^3$; since 5 feet = 60 inches, we get

$$\frac{w(60+\epsilon)}{w(60)} = \frac{C(60+\epsilon)^3}{C(60)^3} = (1+\frac{\epsilon}{60})^3 \ \Rightarrow \ w(60+\epsilon) \approx w(60)\cdot(1+\frac{3\epsilon}{60}) \approx 120\cdot(1+\frac{1}{20}) \approx 126.$$

[Or you can calculate the linearization of $w(h)$ arround $h = 60$ using derivatives, and using the value $w(60)$ to determine C. getting $w(h) \approx 120 + 6(h-60)$

2A-6 $\tan\theta = \frac{\sin\theta}{\cos\theta} \approx \frac{\theta}{1-\theta^2/2} \approx \theta(1+\theta^2//2) \approx \theta$

2A-7 $\frac{\sec x}{\sqrt{1-x^2}} = \frac{1}{\cos x\sqrt{1-x^2}} \approx \frac{1}{(1-\frac{1}{2}x^2)(1-\frac{1}{2}x^2)} \approx \frac{1}{1-x^2} \approx 1+x^2$

2A-8 $\frac{1}{1-x} = \frac{1}{1-(\frac{1}{2}+\Delta x)} = \frac{1}{\frac{1}{2}-\Delta x} = \frac{2}{1-2\Delta x}$

$$\approx \ 2(1+2\Delta x+4(\Delta x)^2) \ \approx \ 2+4(x-\tfrac{1}{2})+8(x-\tfrac{1}{2})^2$$

2A-10 $y = (1+x)^r, y' = r(1+x)^{r-1}, y'' = r(r-1)(1+x)^{r-2}$

Therefore $y(0) = 1, y'(0) = r, y''(0) = r(r-1)$, giving $(1+x)^r \ \approx \ 1 + rx + \frac{r(r-1)}{2}x^2.$

2A-11 $pv^k = c \ \Rightarrow \ p = cv^{-k} = c((v_0+\Delta v)^{-k} = c{v_0}^{-k}(1+\frac{\Delta v}{v_0})^{-k}$

$$\approx \frac{c}{{v_0}^k}(1-k\frac{\Delta v}{v_0} + \frac{k(k+1)}{2}(\frac{\Delta v}{v_0})^2)$$

2A-12 a) $\frac{e^x}{1-x} \approx (1+x+\frac{x^2}{2})(1+x+x^2) \approx 1+2x+\frac{5}{2}x^2$

b) $\dfrac{\ln(1+x)}{xe^x} \approx \dfrac{x - x^2/2}{x(1+x)} = \dfrac{x(1-x/2)}{x(1+x)} = \dfrac{1-x/2}{1+x} \approx (1-x/2)(1-x) \approx 1 - 3x/2$

c) $e^{-x^2} \approx 1 - x^2$ [Substitute into $e^x \approx 1 + x$]

d) $\ln(\cos x) \approx \ln(1 - \dfrac{x^2}{2}) \approx -\dfrac{x^2}{2}$ [since $\ln(1+h) \approx h$]

e) $x \ln x = (1+h)\ln(1+h) \approx (1+h)(h - \dfrac{h^2}{2}) \approx h + \dfrac{h^2}{2} \Rightarrow x \ln x \approx (x-1) + \dfrac{(x-1)^2}{2}$

2A-13 Finding the linear and quadratic approximation

a) $2x$ (both linear and quadratic)

b) $1,\ 1 - 2x^2$

c) $1,\ 1 + x^2/2$ (Use $(1+u)^{-1} \approx 1 - u$ with $u = x^2/2$:

$$\sec x = 1/\cos x \approx 1/(1 - x^2/2) = (1 - x^2/2)^{-1} \approx 1 + x^2/2$$

d) $1,\ 1 + x^2$

e) Use $(1+u)^{-1} \approx 1 - u + u^2$:

$$(a + bx)^{-1} = a^{-1}(1 + (bx/a))^{-1} \approx a^{-1}(1 - bx/a + (bx/a)^2)$$

Linear approximation: $(1/a) - (b/a^2)x$

Quadratic approximation: $(1/a) - (b/a^2)x + (b^2/a^3)x^2$

f) $f(x) = 1/(a + bx)$ so that $f'(1) = -b(a+b)^{-2}$ and $f''(1) = 2b^2/(a+b)^{-3}$. We need to assume that these numbers are defined, in other words that $a + b \neq 0$. Then the linear approximation is

$$1/(a+b) - (b/(a+b)^2)(x-1)$$

and the quadratic approximation is

$$1/(a+b) - (b/(a+b)^2)(x-1) + (b/(a+b)^3)(x-1)^2$$

Method 2: Write

$$1/(a + bx) = 1/(a + b + b(x-1))$$

Then use the expansion of problem (e) with $a+b$ in place of a and b in place of b and $(x-1)$ in place of x. The requirement $a \neq 0$ in (e) corresponds to the restriction $a + b \neq 0$ in (f).

2A-15 $f(x) = \cos(3x)$, $\quad f'(x) = -3\sin(3x)$, $\quad f''(x) = -9\cos(3x)$. Thus,

$$f(0) = 1, \quad f(\pi/6) = \cos(\pi/2) = 0, \quad f(\pi/3) = \cos\pi = -1$$
$$f'(0) = -3\sin 0 = 0, \quad f'(\pi/6) = -3\sin(\pi/2) = -3, \quad f'(\pi/3) = -3\sin\pi = 0$$
$$f''(0) = -9, \quad f''(\pi/6) = 0, \quad f''(\pi/3) = 9$$

Using these values, the linear and quadratic approximations are respectively:

$$\begin{aligned} \text{for } x \approx 0: &\quad f(x) \approx 1 \text{ and } f(x) \approx 1 - (9/2)x^2 \\ \text{for } x \approx \pi/6: &\quad \text{both are } f(x) \approx -3(x - \pi/6) \\ \text{for } x \approx \pi/3: &\quad f(x) \approx -1 \text{ and } f(x) \approx -1 + (9/2)(x - \pi/3)^2 \end{aligned}$$

2A-16 a) The law of cosines says that for a triangle with sides a, b, and c, with θ opposite the side of length c,

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

Apply it to one of the n triangles with vertex at the origin: $a = b = 1$ and $\theta = 2\pi/n$. So the formula is

$$c = \sqrt{2 - 2\cos(2\pi/n)}$$

b) The perimeter is $n\sqrt{2 - 2\cos(2\pi/n)}$. The quadratic approximation to $\cos\theta$ near 0 is

$$\cos\theta \approx 1 - \theta^2/2$$

Therefore, as $n \to \infty$ and $\theta = 2\pi/n \to 0$,

$$n\sqrt{2 - 2\cos(2\pi/n)} \approx n\sqrt{2 - 2(1 - (1/2)(2\pi/n)^2)} = n\sqrt{(2\pi/n)^2} = n(2\pi/n) = 2\pi$$

In other words,

$$\lim_{n\to\infty} n\sqrt{2 - 2\cos(2\pi/n)} = 2\pi,$$

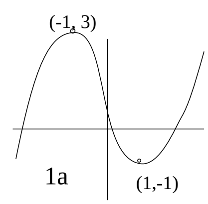
the circumference of the circle of radius 1.

2B. Curve Sketching

2B-1 a) $y = x^3 - 3x + 1$, $y' = 3x^2 - 3 = 3(x-1)(x+1)$. $y' = 0 \implies x = \pm 1$.

Endpoint values: $y \to -\infty$ as $x \to -\infty$, and $y \to \infty$ as $x \to \infty$.

Critical values: $y(-1) = 3$, $y(1) = -1$.



Increasing on: $-\infty < x < -1$, $1 < x < \infty$.

Decreasing on: $-1 < x < 1$.

Graph: $(-\infty, -\infty) \nearrow (-1, 3) \searrow (1, -1) \nearrow (\infty, \infty)$, crossing the x-axis three times.

b) $y = x^4 - 4x + 1$, $y' = x^3 - 4$. $y' = 0 \implies x = 4^{1/3}$.

Increasing on: $4^{1/3} < x < \infty$; decreasing on: $-\infty < x < 4^{1/3}$.

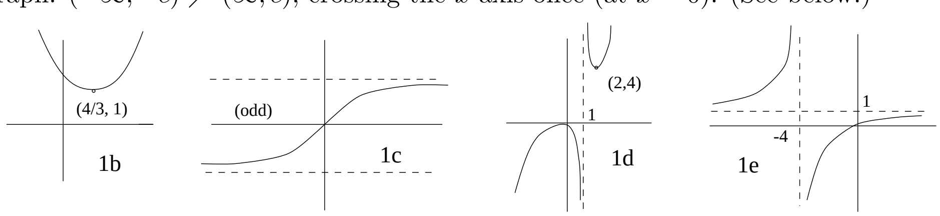
Endpoint values: $y \to \infty$ as $x \to \pm\infty$; critical value: $y(4^{1/3}) = 1$.

Graph: $(-\infty, \infty) \searrow (4^{1//3}, 1) \nearrow (\infty, \infty)$, never crossing the x-axis. (See below.)

c) $y'(x) = 1/(1 + x^2)$ and $y(0) = 0$. By inspection, $y' > 0$ for all x, hence always increasing.

Endpoint values: $y \to c$ as $x \to \infty$ and by symmetry $y \to -c$ as $x \to -\infty$. (But it is not clear at this point in the course whether $c = \infty$ or some finite value. It turns out (in Lecture 26) that $y \to c = \pi/2$.

Graph: $(-\infty, -c) \nearrow (\infty, c)$, crossing the x-axis once (at $x = 0$). (See below.)



d) $y = x^2/(x-1)$, $y' = (2x(x-1)-x^2)/(x-1)^2 = (x^2-2x)/(x-1)^2 = (x-2)x/(x-1)^2$.

Endpoint values: $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$.

Singular values: $y(1^+) = +\infty$ and $y(1^-) = -\infty$.

Critical values: $y(0) = 0$ and $y(2) = 4$.

New feature: Pay attention to sign changes in the denominator of y'.

Increasing on: $-\infty < x < 0$ and $2 < x < \infty$

Decreasing on: $0 < x < 1$ and $1 < x < 2$

Graph: $(-\infty,-\infty) \nearrow (0,0) \searrow (1,-\infty) \uparrow (1,\infty) \searrow (2,4) \nearrow (\infty,\infty)$, crossing the x-axis once (at $x = 0$).

Commentary on singularities: Look out for sign changes both where y' is zero and also where y' is undefined: $y' = 0$ indicates a possible sign change in the numerator and y' undefined indicates a possible sign change in the denominator. In this case there was no sign change in y' at $x = 1$, but there would have been a sign change, if there had been an odd power of $(x-1)$ in the denominator.

e) $y = x/(x+4)$, $y' = ((x+4)-x)/(x+4)^2 = 4/(x+4)^2$. No critical points.

Endpoint values: $y \to 1$ as $x \to \pm\infty$.

Increasing on: $-4 < x < \infty$.

Decreasing on: $-\infty < x < -4$.

Singular values: $y(-4^+) = -\infty$, $y(-4^-) = +\infty$.

Graph: $(-\infty, 1) \nearrow (-4,\infty) \downarrow (-4,-\infty) \nearrow (\infty, 1)$, crossing the x-axis once (at $x = 0$).

f) $y = \sqrt{x+1}/(x-3)$, $y' = -(1/2)(x+5)(x+1)^{-1/2}(x-3)^{-2}$ No critical points because $x = -5$ is outside of the domain of definition, $x \geq -1$.

Endpoint values: $y(-1) = 0$, and as $x \to \infty$,

$$y = \frac{1 + \frac{1}{x}}{\sqrt{x} - \frac{3}{\sqrt{x}}} \to \frac{1}{\infty} = 0$$

Singular values: $y(3^+) = +\infty$, $y(3^-) = -\infty$.

Increasing on: nowhere

Decreasing on: $-1 < x < 3$ and $3 < x < \infty$.

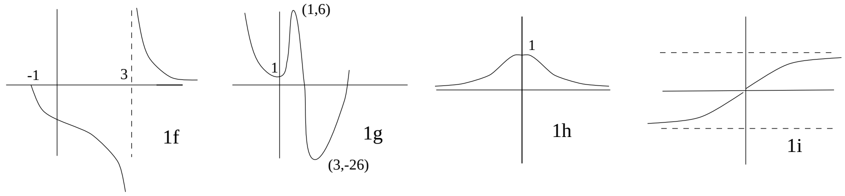
Graph: $(-1,0) \searrow (3,-\infty) \uparrow (3,\infty) \searrow (\infty,0)$, crossing the x-axis once (at $x = -1$).

g) $y = 3x^4 - 16x^3 + 18x^2 + 1$, $y' = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$. $y' = 0 \implies x = 0,\ 1,\ 3$.

Endpoint values: $y \to \infty$ as $x \to \pm\infty$.

Critical values: $y(0) = 1$, $y(1) = 6$, and $y(3) = -188$.

Increasing on: $0 < x < 1$ and $3 < x < \infty$.



Decreasing on: $-\infty < x < 0$ and $1 < x < 3$.

Graph: $(-\infty, \infty) \searrow (0, 1) \nearrow (1, 6) \searrow (3, -188) \nearrow (\infty, \infty)$, crossing the x-axis once.

h) $y = e^{-x^2}$, $y' = -2xe^{-x^2}$. $y' = 0 \implies x = 0$.

Endpoint values: $y \to 0$ as $x \to \pm\infty$.

Critical value: $y(0) = 1$.

Increasing on: $-\infty < x < 0$

Decreasing on: $0 < x < \infty$

Graph: $(-\infty, 0) \nearrow (0, 1) \searrow (\infty, 0)$, never crossing the x-axis. (The function is even.)

i) $y' = e^{-x^2}$ and $y(0) = 0$. Because y' is even and $y(0) = 0$, y is odd. No critical points.

Endpoint values: $y \to c$ as $x \to \infty$ and by symmetry $y \to -c$ as $x \to -\infty$. It is not clear at this point in the course whether c is finite or infinite. But we will be able to show that c is finite when we discuss improper integrals in Unit 6. (Using a trick with iterated integrals, a subject in 18.02, one can show that $c = \sqrt{\pi}/2$.)

Graph: $(-\infty, -c) \nearrow (\infty, c)$, crossing the x-axis once (at $x = 0$).

2B-2 a) One inflection point at $x = 0$. ($y'' = 6x$)

b) No inflection points. $y'' = 3x^2$, so the function is convex. $x = 0$ is not a point of inflection because $y'' > 0$ on both sides of $x = 0$.

c) Inflection point at $x = 0$. ($y'' = -2x/(1 + x^2)^2$)

d) No inflection points. Reasoning: $y'' = 2/(x - 1)^3$. Thus $y'' > 0$ and the function is concave up when $x > 1$, and $y'' < 0$ and the function is concave down when $x < 1$. But $x = 1$ is not called an inflection point because the function is not continuous there. In fact, $x = 1$ is a singular point.

e) No inflection points. $y'' = -8/(x + 1)^3$. As in part (d) there is a sign change in y'', but at a singular point not an inflection point.

f) $y'' = -(1/2)[(x+1)(x-3) - (1/2)(x+5)(x-3) - 2(x+5)(x+1)](x+1)^{-3/2}(x-3)^3$

$= -(1/2)[-(3/2)x^2 - 15x - 11/2](x + 1)^{-3/2}(x - 3)^3$

Therefore there are two inflection points, $x = (-30 \pm \sqrt{768})/6$, $\approx 9.6, .38$.

g) $y'' = 12(3x^2 - 8x + 36)$. Therefore there are no inflection points. The quadratic equation has no real roots.

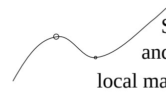
h) $y'' = (-2 + 4x^2)e^{-x^2}$. Therefore there are two inflection points at $x = \pm 1/\sqrt{2}$.

i) One inflection point at $x = 0$. ($y'' = -2xe^{-x^2}$)

2B-3 a) $y' = 3x^2 + 2ax + b$. The roots of the quadratic polynomial are distinct real numbers if the discriminant is positive. (The *discriminant* is defined as the number under the square root in the quadratic formula.) Therefore there are distinct real roots if and only if

$$(2a)^2 - 4(3)b > 0, \qquad \text{or} \qquad a^2 - 3b > 0.$$

From the picture, since $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$, the larger root of $3x^2 + 2ax + b = 0$ (with the plus sign in the quadratic formula) must be the local min, and the smaller root must be the local max.



b) $y'' = 6x + 2a$, so the inflection point is at $-a/3$. Therefore the condition $y' < 0$ at the inflection point is

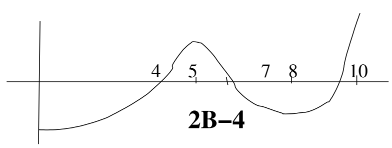
$$y'(-a/3) = 3(-a/3)^2 + 2a(-a/3) + b = -a^2/3 + b < 0,$$

which is the same as

$$a^2 - 3b > 0.$$

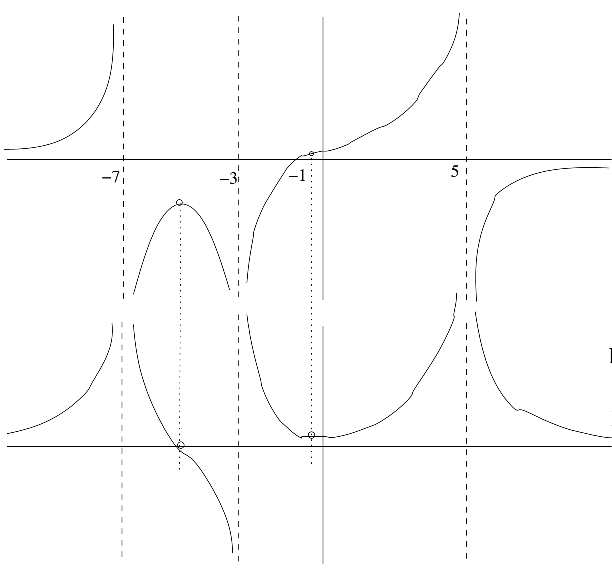
If $y' < 0$ at some point x_0, then the function is decreasing at that point. But $y \to \infty$ as $x \to \infty$, so there must be a local minimum at a point $x > x_0$. Similarly, since $y \to -\infty$ as $x \to -\infty$, there must be a local maximum at a point $x < x_0$.

Comment: We evaluate y' at the inflection point of y ($x = -a/3$) since we are trying to decide (cf. part (b)) whether y' is ever negative. To do this, we find the minimum of y' (which occurs where $y'' = 0$).



2B–4

Max is at x = 5 or x = 10;
Min is at x = 0 or x = 8.



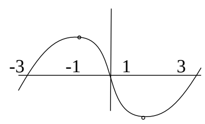
2B–5

2B-6 a) Try $y' = (x+1)(x-1) = x^2 - 1$. Then $y = x^3/3 - x + c$. The constant c won't matter so set $c = 0$. It's also more convenient to multiply by 3:

$$y = x^3 - 3x$$

b) This is an odd function with local min and max: $y(1) = -2$ and $y(-1) = 2$. The endpoints values are $y(3) = 18$ and $y(-3) = -18$. It is very steep: $y'(3) = 8$.

c)



2B-7 a) $f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

If y increasing then $\left.\begin{array}{l}\Delta y > 0 \Rightarrow \Delta x > 0 \\ \Delta y < 0 \Rightarrow \Delta x < 0\end{array}\right\}$. So in both cases $\frac{\Delta y}{\Delta x} > 0$.

Therefore, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \geq 0$.

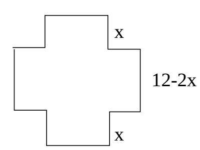
b) Proof breaks down at the last step. Namely, $\frac{\Delta y}{\Delta x} > 0$ doesn't imply $\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta y} > 0$

[Limits don't preserve strict inequalities, only weak ones. For example, $u^2 > 0$ for $u \neq 0$, but $\lim_{u \to 0} u^2 = 0 \geq 0$, not > 0.]

Counterexample: $f(x) = x^3$ is increasing for all x, but $f'(0) = 0$.

c) Use $f(a) \geq f(x)$ to show that $\lim_{\Delta x \to 0^+} \Delta y/\Delta x \leq 0$ and $\lim_{\Delta x \to 0^-} \Delta y/\Delta x \geq 0$. Since the left and right limits are equal, the derivative must be zero.

2C. Max-min problems

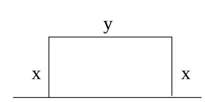


2C-1 The base of the box has sidelength $12 - 2x$ and the height is x, so the volume is $V = x(12 - 2x)^2$.

At the endpoints $x = 0$ and $x = 6$, the volume is 0, so the maximum must occur in between at a critical point.

$$V' = (12 - 2x)^2 + x(2)(12 - 2x)(-2) = (12 - 2x)(12 - 2x - 4x) = (12 - 2x)(12 - 6x).$$

It follows that $V' = 0$ when $x = 6$ or $x = 2$. At the endpoints $x = 0$ and $x = 6$ the volume is 0, so the maximum occurs when $x = 2$.



2C-2 We want to minimize the fence length $L = 2x + y$, where the variables x and y are related by $xy = A = 20,000$.

Choosing x as the independent variable, we have $y = A/x$, so that $L = 2x + A/x$. At the endpoints $x = 0$ and $x = \infty$ (it's a long barn), we get $L = \infty$, so the minimumof L must occur at a critical point.

$$L' = 2 - \frac{A}{x^2}; \quad L = 0 \implies x^2 = \frac{A}{2} = 10,000 \implies x = 100 \text{ feet}$$

2C-3 We have $y = (a-x)/2$, so $xy = x(a-x)/2$. At the endpoints $x = 0$ and $x = a$, the product xy is zero (and beyond it is negative). Therefore, the maximum occurs at a critical point. Taking the derivative,

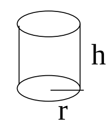
$$\frac{d}{dx}\frac{x(a-x)}{2} = \frac{a-2x}{2}; \quad \text{this is 0 when } x = a/2.$$

2C-4 If the length is y and the cross-section is a square with sidelength x, then $4x+y = 108$. Therxefore the volume is $V = x^2y = 108x^2 - 4x^3$. Find the critical points:

$$(108x^2 - 4x^3)' = 216x - 12x^2 = 0 \implies x = 18 \text{ or } x = 0.$$

The critical point $x = 18$ (3/2 ft.) corresponds to the length $y = 36$ (3 ft.), giving therefore a volume of $(3/2)^2(3) = 27/4 = 6.75$ cubic feet.

The endpoints are $x = 0$, which gives zero volume, and when $x = y$, i.e., $x = 9/5$ feet, which gives a volume of $(9/5)^3$ cubic feet, which is less than 6 cubic feet. So the critical point gives the maximum volume.



2C-5 We let r = radius of bottom and h = height, then the volume is $V = \pi r^2 h$, and the area is $A = \pi r^2 + 2\pi rh$.

Using r as the independent variable, we have using the above formulas,

$$h = \frac{A - \pi r^2}{2\pi r}, \qquad V = \pi r^2 h = \left(\frac{A}{2}r - \frac{\pi}{2}r^3\right); \qquad \frac{dV}{dr} = \frac{A}{2} - \frac{3\pi}{2}r^2.$$

Therefore, $dV/dr = 0$ implies $A = 3\pi r^2$, from which $h = \dfrac{A - \pi r^2}{2\pi r} = r$.

Checking the endpoints, at one $h = 0$ and $V = 0$; at the other, $\lim_{r\to 0+} V = 0$ (using the expression above for V in terms of r); thus the critical point must occur at a maximum.

(Another way to do this problem is to use implicit differentiation with respect to r. Briefly, since A is fixed, $dA/dr = 0$, and therefore

$$\frac{dA}{dr} = 2\pi r + 2\pi h + 2\pi rh' = 0 \implies h' = -\frac{r+h}{r};$$
$$\frac{dV}{dr} = 2\pi rh + \pi r^2 h' = 2\pi rh - \pi r(r+h) = \pi r(h-r).$$

It follows that $V' = 0$ when $r = h$ or $r = 0$, and the latter is a rejected endpoint.

2C-6 To get max and min of $y = x(x+1)(x-1) = x^3 - x$, first find the critical points:

$$y' = 3x^2 - 1 = 0 \quad \text{if} \quad x = \pm\frac{1}{\sqrt{3}};$$

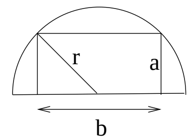
$$y(\frac{1}{\sqrt{3}}) = \frac{1}{\sqrt{3}}(-\frac{2}{3}) = \frac{-2}{3\sqrt{3}}, \text{ rel. min.} \qquad y(-\frac{1}{\sqrt{3}}) = -\frac{1}{\sqrt{3}}(-\frac{2}{3}) = \frac{2}{3\sqrt{3}}, \text{ rel. max.}$$

Check endpoints: $y(2) = 6 \Rightarrow 2$ is absolute max.; $y(-2) = -6 \Rightarrow -2$ is absolute min.

(This is an *endpoint problem.* The endpoints should be tested unless the physical or geometric picture already makes clear whether the max or min occurs at an endpoint.)

2C-7 Let r be the radius, which is fixed. Then the height a of the rectangle is in the interval $0 \le a \le r$. Since $b = 2\sqrt{r^2 - a^2}$, the area A is given in terms of a by

$$A = 2a\sqrt{r^2 - a^2}.$$



The value of A at the endpoints $a = 0$ and $a = r$ is zero, so the maximum occurs at a critical point in between.

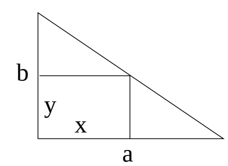
$$\frac{dA}{da} = 2\sqrt{r^2 - a^2} - \frac{2a^2}{\sqrt{r^2 - a^2}} = \frac{2(r^2 - a^2) - 2a^2}{\sqrt{r^2 - a^2}}$$

Thus $dA/da = 0$ implies $2r^2 = a^2$, from which we get $\quad a = \dfrac{r}{\sqrt{2}}, \quad b = r\sqrt{2}.$

(We use the positive square root since $a \ge 0$. Note that $b = 2a$ and $A = r^2$.)

2C-8 a) Letting a and b be the two legs and x and y the sides of the rectangle, we have $y = -(b/a)(x - a)$ and the area $A = xy = (b/a)x(a - x)$. The area is zero at the two ends $x = 0$ and $x = a$, so the maximum occurs in between at a critical point:

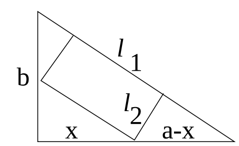
$$A' = (b/a)((a - x) - x); \quad = 0 \text{ if } x = a/2.$$



Thus $y = (b/a)(a - x) = b/2$ and $A = ab/4$.

b) This time let x be the point shown on the accompanying figure; using similar triangles, the sides of the rectangle are

$$\ell_1 = \frac{x}{a}\sqrt{a^2 + b^2} \text{ and } \ell_2 = \frac{b}{\sqrt{a^2 + b^2}}(a - x)$$



Therefore the area is

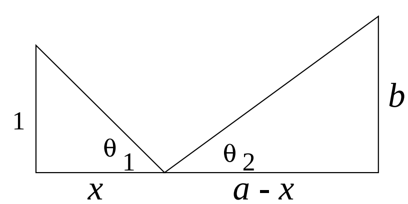
$$A = \ell_1\ell_2 = (b/a)x(a - x)$$

This is the same formula for area as in part (a), so the largest area is the same, occurring when $x = a/2$, and the two maximal rectangles both have the same area; they have different dimensions though, since in the present case, one side length is half the hypotenuse:

$$\ell_1 = \sqrt{a^2 + b^2}/2 \text{ and } \ell_2 = ab/2\sqrt{a^2 + b^2}.$$

2C-9 The distance is

$$L = \sqrt{x^2 + 1} + \sqrt{(a - x)^2 + b^2}$$



The endpoint values are $x \to \pm\infty$, for $L \to \infty$, so the minimum value is at a critical point.

$$L' = \frac{x}{\sqrt{x^2 + 1}} - \frac{a - x}{\sqrt{(a - x)^2 + b^2}} = \frac{x\sqrt{(a - x)^2 + b^2} - (a - x)\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}\sqrt{(a - x)^2 + b^2}}$$

Thus $L' = 0$ implies (after squaring both sides),

$$x^2((a - x)^2 + b^2) = (a - x)^2(x^2 + 1), \quad \text{or} \quad x^2b^2 = (a - x)^2 \quad \text{or} \quad bx = (a - x);$$

we used the positive square roots since both sides must be positive. Rewriting the above,

$$\frac{b}{a-x} = \frac{1}{x}, \qquad \text{or} \qquad \tan\theta_1 = \tan\theta_2.$$

Thus $\theta_1 = \theta_2$: the angle of incidence equals the angle of reflection.

2C-10 The total time is

$$T = \frac{\sqrt{100^2 + x^2}}{5} + \frac{\sqrt{100^2 + (a-x)^2}}{2}$$

As $x \to \pm\infty$, $T \to \infty$, so the minimum value will be at a critical point.

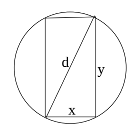
$$T' = \frac{x}{5\sqrt{100^2 + x^2}} - \frac{(a-x)}{2\sqrt{100^2 + (a-x)^2}} = \frac{\sin\alpha}{5} - \frac{\sin\beta}{2}.$$

Therefore, if $T' = 0$, it follows that

$$\frac{\sin\alpha}{5} = \frac{\sin\beta}{2} \quad \text{or} \quad \frac{\sin\alpha}{\sin\beta} = \frac{5}{2}.$$

2C-11 Use implicit differentiation:

$x^2 + y^2 = d^2 \implies 2x + 2yy' = 0 \implies y' = -x/y.$



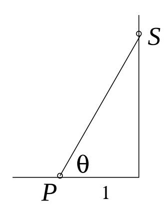
We want to maximize xy^3. At the endpoints $x = 0$ and $y = 0$, the strength is zero, so there is a maximum at a critical point. Differentiating,

$$0 = (xy^3)' = y^3 + 3xy^2y' = y^3 + 3xy^2(-x/y) = y^3 - 3x^2y$$

Dividing by x^3,

$$(y/x)^3 - 3(y/x) = 0 \implies (y/x)^2 = 3 \implies y/x = \sqrt{3}.$$

2C-12 The intensity is proportional to



$$y = \frac{\sin\theta}{1+x^2} = \frac{x/\sqrt{1+x^2}}{1+x^2} = x(1+x^2)^{-3/2}$$

Endpoints: $y(0) = 0$ and $y \to 0$ as $x \to \infty$, so the maximum will be at a critical point. Critical points satisfy

$$y' = (1-2x^2)(1+x^2)^{-5/2} = 0 \implies 1 - 2x^2 = 0 \implies x = 1/\sqrt{2}$$

The best height is $1/\sqrt{2}$ feet above the desk. (It's not worth it. Use a desk lamp.)

2C-13 a) Let p denote the price in dollars. Then there will be $100 + (2/5)(200 - p)$ passengers. Therefore the total revenue is

$$R = p(100 + (2/5)(200 - p)) = p(180 - (2/5)p)$$

At the "ends" zero price $p = 0$, and no passengers $p = (5/2)180 = 450$, the revenue is zero. So the maximum occurs in between at a critical point.

$$R' = (180 - (2/5)p) - (2/5)p = 180 - (4/5)p = 0 \implies p = (5/4)180 = \$225$$

b)

$$P = xp - x(10 - x/10^5) \text{ with } x = 10^5(10 - p/2)$$

Therefore, the profit is cents is

$$P = 10^5(10 - p/2)(p - 10 + (10 - p/2)) = 10^5(10 - p/2)(p/2) = (10^5/4)p(20 - p)$$

$$\frac{dP}{dp} = (10^5/2)(10 - p)$$

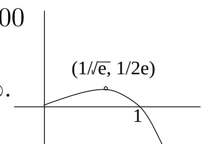
The critical point at $p = 10$. This is $x = 10^5(10 - 5) = 5 \times 10^5$ kilowatt hours, which is within the range available to the utility company. The function P has second derivative $-10^5/2$, so it is concave down and the critical point must be the maximum. (This is one of those cases where checking the second derivative is easier than checking the endpoints.)

Alternatively, the endpoint values are:

$$x = 0 \implies 10^5(10 - p/2) = 0 \implies p = 20 \implies P = 0.$$

$$\begin{aligned} x = 8 \times 10^5 &\implies 8 \times 10^5 = 10^5(10 - p/2) \\ &\implies 10 - p/2 = 8 \implies p = 4 \\ &\implies P = (10^5/4)4(20 - 4) = 16 \times 10^5 \text{cents} = \$160,000 \end{aligned}$$

The profit at the crit. pt. was $(10^5/4)10(20 - 10) = 2.5 \times 10^6$cents $= \$250,000$



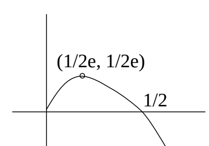
14a

2C-14 a) Endpoints: $y = -x^2 \ln(x) \to 0$ as $x \to 0^+$ and $y \to -\infty$ as $x \to \infty$.

Critical points: $y' = -2x \ln x - x = 0 \implies \ln x = -1/2 \implies x = 1/\sqrt{e}$.

Critical value: $y(1/\sqrt{e}) = 1/2e$.

Maximum value: $1/2e$, attained when $x = 1/\sqrt{e}$. (min is not attained)



14b

b) Endpoints: $y = -x \ln(2x) \to 0$ as $x \to 0^+$ and $y \to -\infty$ as $x \to \infty$.

Critical points: $y' = -\ln(2x) - 1 = 0 \implies x = 1/2e$.

Critical value: $y(1/2e) = -(1/2e)\ln(1/e) = 1/2e$.

Maximum value: $1/2e$, attained at $x = 1/2e$. (min is not attained)

2C-15 No minimum. The derivative is $-xe^{-x} < 0$, so the function decreases. (Not needed here, but it will follows from 2G-8 or from L'Hospital's rule in E31 that $xe^{-x} \to 0$ as $x \to \infty$.)

2D. More Max-min Problems

2D-3 The milk will be added at some time t_1, such that $0 \leq t_1 \leq 10$. In the interval $0 \leq t < t_1$ the temperature is

$$y(t) = (100 - 20)e^{-(t-0)/10} + 20 = 80e^{-t/10} + 20$$

Therefore,

$$T_1 = y(t_1^-) = 80e^{-t_1/10} + 20$$

We are adding milk at a temperature $T_2 = 5$, so the temperature as we start the second interval of cooling is

$$\frac{9}{10}T_1 + \frac{1}{10}T_2 = 72e^{-t_1/10} + 18 + \frac{1}{2}$$

Let $Y(t)$ be the coffee temperature in the interval $t_1 \leq t \leq 10$. We have just calculated $Y(t_1)$, so

$$5Y(t) = (Y(t_1) - 20)e^{-(t-t_1)/10} + 20 = (72e^{-t_1/10} - 1.5)e^{-(t-t_1)/10} + 20$$

The final temperature is

$$T = Y(10) = (72e^{-t_1/10} - 1.5)e^{-(10-t_1)/10} + 20 = e^{-1}\left(72 - (1.5)e^{t_1/10}\right) + 20$$

We want to maximize this temperature, so we look for critical points:

$$\frac{dT}{dt_1} = -(1.5/10e)e^{t_1/10} < 0$$

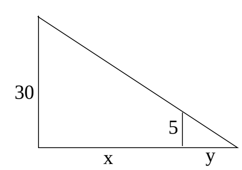
Therefore the function $T(t_1)$ is decreasing and its maximum occurs at the left endpoint: $t_1 = 0$.

Conclusion: The coffee will be hottest if you put the milk in as soon as possible.

2E. Related Rates

2E-1 The distance from robot to the point on the ground directly below the street lamp is $x = 20t$. Therefore, $x' = 20$.

$$\frac{x+y}{30} = \frac{y}{5} \quad \text{(similar triangles)}$$



Therefore,

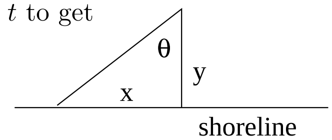
$$(x' + y')/30 = y'/5 \implies y' = 4 \text{ and } (x+y)' = 24$$

The tip of the shadow is moving at 24 feet per second and the length of the shadow is increasing at 4 feet per second.

2E-2

$$\tan\theta = \frac{x}{4} \qquad \text{and } d\theta/dt = 3(2\pi) = 6\pi$$

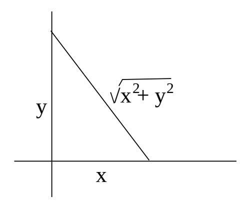
with t is measured in minutes and θ measured in radians. The light makes an angle of 60° with the shore when θ is 30° or $\theta = \pi/6$. Differentiate with respect to t to get



$$(\sec^2\theta)(d\theta/dt) = (1/4)(dx/dt)$$

Since $\sec^2(\pi/6) = 4/3$, we get $dx/dt = 32\pi$ miles per minute.

2E-3 The distance is $x = 10$, $y = 15$, $x' = 30$ and $y' = 30$. Therefore,



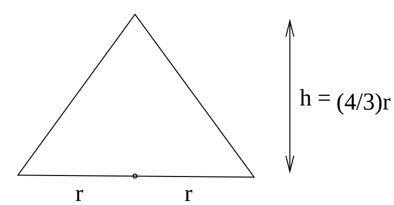
$$\begin{aligned}\left((x^2+y^2)^{1/2}\right)' &= (1/2)(2xx' + 2yy')(x^2+y^2)^{-1/2}\\ &= (10(30) + 15(30))/\sqrt{10^2+15^2}\\ &= 150/\sqrt{13}\text{miles per hour}\end{aligned}$$

2E-4 $V = (\pi/3)r^2h$ and $2r = d = (3/2)h$ implies $h = (4/3)r$.

Therefore,

$$V = (\pi/3)r^2h = (4\pi/9)r^3$$

Moreover, $dV/dt = 12$, hence



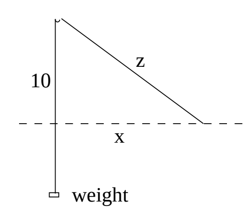
$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = \left(\frac{d}{dr}(4\pi/9)r^3\right)\frac{dr}{dt} = (4\pi/3)r^2(12) = 16\pi r^2.$$

When $h = 2$, $r = 3/2$, so that $\dfrac{dV}{dt} = 36\pi\text{m}^3/\text{minute}$.

2E-5 The information is

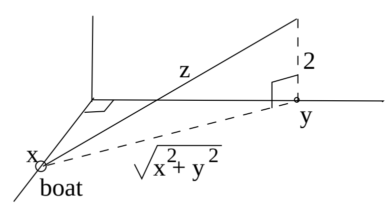
$$x^2 + 10^2 = z^2, \quad z' = 4$$

We want to evaluate x' at $x = 20$. (Derivatives are with respect to time.) Thus



$$2xx' = 2zz' \text{ and } z^2 = 20^2 + 10^2 = 500$$

Therefore,



$$x' = (zz')/x = 4\sqrt{500}/20 = 2\sqrt{5}$$

2E-6 $x' = 50$ and $y' = 400$ and

$$z^2 = x^2 + y^2 + 2^2$$

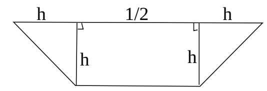
The problem is to evaluate z' when $x = 50$ and $y = 400$. Thus

$$2zz' = 2xx' + 2yy' \implies z' = (xx' + yy')/z$$

and $z = \sqrt{50^2 + 400^2 + 4} = \sqrt{162504}$. So $z' = 162500/\sqrt{162504} \approx 403$mph.

(The fact that the plane is 2 miles up rather than at sea level changes the answer by only about 4/1000. Even the boat speed only affects the answer by about 3 miles per hour.)

2E-7 $V = 4(h^2 + h/2)$, $V' = 1$. To evaluate h' at $h = 1/2$,

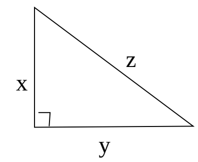


$$1 = V' = 8hh' + 2h' = 8(1/2)h' + 2h' = 6h'$$

Therefore,

$$h' = 1/6 \text{ meters per second}$$

2E-8 $x' = 60$, $y' = 50$ and $x = 60 + 60t$, $y = 50t$. Noon is $t = 0$ and t is measured in hours. To find the time when $z = \sqrt{x^2 + y^2}$ is smallest, we may as well minimize $z^2 = x^2 + y^2$. We know that there will be a minimum at a critical point because when $t \to \pm\infty$ the distance tends to infinity. Taking the derivative with respect to t, the critical points satisfy



$$2xx' + 2yy' = 0$$

This equation says

$$2((60 + 60t)60 + (50t)50) = 0 \implies (60^2 + 50^2)t = -60^2$$

Hence

$$t = -36/61 \approx -35\text{min}$$

The ships were closest at around 11 : 25 am.

2E-9 $dy/dt = 2(x - 1)dx/dt$. Notice that in the range $x < 1$, $x - 1$ is negative and so $(x - 1) = -\sqrt{y}$. Therefore,

$$dx/dt = (1/2(x - 1))(dy/dt) = -(1/2\sqrt{y})(dy/dt) = +(\sqrt{y})(1 - y)/2 = 1/4\sqrt{2}$$

Method 2: Doing this directly turns out to be faster:

$$x = 1 - \sqrt{y} \implies dx/dt = 1 - (1/2)y^{-1/2}dy/dt$$

and the rest is as before.

2E-10 $r = Ct^{1/2}$. The implicit assumption is that the volume of oil is constant:

$$\pi r^2 T = V \text{ or } r^2 T = (V/\pi) = \text{const}$$

Therefore, differentiating with respect to time t,

$$(r^2T)' = 2rr'T + r^2T' = 0 \implies T' = -2r'/r$$

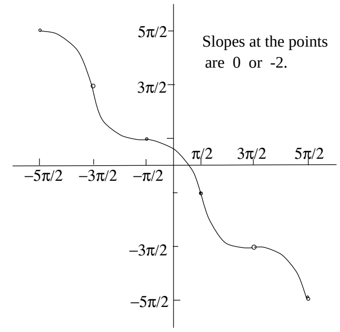
But $r' = (1/2)Ct^{-1/2}$, so that $r'/r = 1/2t$. Therefore

$$T' = -1/t$$

(Although we only know the rate of change of r up to a constant of proportionality, we can compute the absolute rate of change of T.)

2F. Locating zeros; Newton's method

2F-1 a) $y' = -\sin x - 1 \leq 0$. Also, $y' < 0$ except at a discrete list of points (where $\sin x = -1$). Therefore y is strictly decreasing, that is, $x_1 < x_2 \implies y(x_1) < y(x_2)$. Thus y crosses zero only once.



Upper and lower bounds for z such that $y(z) = 0$:

$y(0) = 1$ and $y(\pi/2) = -\pi/2$. Therefore, $0 < z < \pi/2$.

b) $x_{n+1} = x_n - (\cos x_n - x_n)/(\sin x_n + 1)$

$$x_1 = 1, \qquad x_2 = .750363868,$$
$$x_3 = .739112891, \quad x_4 = .739085133$$

Accurate to three decimals at x_3, the second step. Answer .739.

c) Fixed point method takes 53 steps to stabilize at .739085133. Newton's method takes only three steps to get to 9 digits of accuracy. (See x_4.)

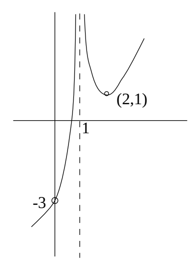
2F-2

$$y = 2x - 4 + \frac{1}{(x-1)^2} \qquad -\infty < x < \infty$$

$$y' = 2 - \frac{2}{(x-1)^3} = \frac{2((x-1)^3 - 1)}{(x-1)^3}$$

$y' = 0$ implies $(x-1)^3 = 1$, which implies $x - 1 = 1$ and hence that $x = 2$. The sign changes of y' are at the critical point $x = 2$ and at the singularity $x = 1$. For $x < 1$, the numerator and denominator are negative, so $y' > 0$. For $1 < x < 2$, the numerator is still negative, but the denominator is positive, so $y' < 0$. For $2 < x$, both numerator and denominator are positive, so $y' > 0$.

$$y'' = \frac{6}{(x-1)^4}$$



Therefore, $y'' > 0$ for $x \neq 1$.

Critical value: $y(2) = 1$

Singular values: $y(1^-) = y(1^+) = \infty$

Endpoint values: $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$.

Conclusion: The function increases from $-\infty$ to ∞ on the interval $(-\infty, 1)$. Therefore, the function vanishes exactly once in this interval. The function decreases from ∞ to 1 on the interval $(1, 2)$ and increases from 1 to ∞ on the interval $(2, \infty)$. Therefore, the function does not vanish at all in the interval $(1, \infty)$. Finally, the function is concave up on the intervals $(-\infty, 1)$ and $(1, \infty)$

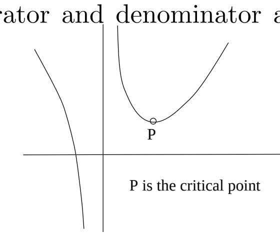
2F-3

$$y' = 2x - x^{-2} = \frac{2x^3 - 1}{x^2}$$

Therefore $y' = 0$ implies $x^3 = 1/2$ or $x = 2^{-1/3}$. Moreover, $y' > 0$ when $x > 2^{-1/3}$, and $y' < 0$ when $x < 2^{-1/3}$ and $x \neq 0$. The sign does not change across the singular point $x = 0$ because the power in the denominator is even. *(continued →)*

$$y'' = 2 + 2x^{-3} = \frac{2(x^3 + 1)}{x^3}$$

Therefore $y'' = 0$ implies $x^3 = -1$, or $x = -1$. Keeping track of the sign change in the denominator as well as the numerator we have that $y'' > 0$ when $x > 0$ and $y'' < 0$ when $-1 < x < 0$. Finally, $y'' > 0$ when $x < -1$, and both numerator and denominator are negative.



Critical value: $y(2^{-1/3}) = 2^{-2/3} + 2^{1/3} \approx 1.9$

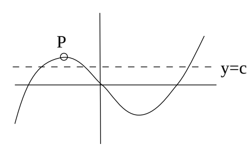
Singular value: $y(0^+) = +\infty$ and $y(0^-) = -\infty$

Endpoint values: $y \to \infty$ as $x \to \pm\infty$

Conclusions: The function decreases from ∞ to $-\infty$ in the interval $(-\infty, 0)$. Therefore it vanishes exactly once in this interval. It jumps to ∞ at 0 and decreases from ∞ to $2^{-2/3} + 2^{1/3}$ in the interval $(0, 2^{-1/3})$. Finally it increases from $2^{-2/3} + 2^{1/3}$ to ∞ in the interval $(2^{-1/3}, \infty)$. Thus it does not vanish on the interval $(0, \infty)$. The function is concave up in the intervals $(-\infty, -1)$ and $(0, \infty)$ and concave down in the interval $(-1, 0)$, with an inflection point at -1.

2F-4 From the graph, $x^5 - x - c = 0$ has three roots for any small value of c. The value of c gets too large if it exceeds the local maximum of $x^5 - x$ labelled. To calculate that local maximum, consider $y' = 5x^4 - 1 = 0$, with solutions $x = \pm 5^{-1/4}$. The local maximum is at $x = -5^{-1/4}$ and the value is

$$(-5^{-1/4})^5 - (-5^{-1/4}) = 5^{-1/4} - 5^{-5/4} \approx .535$$



Since $.535 > 1/2$, there are three roots.

2F-5 a) Answer: $x_1 = \pm 1/\sqrt{3}$. $f(x) = x - x^3$, so $f'(x) = 1 - 3x^2$ and

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

So x_2 is undefined if $f'(x_1) = 0$, that is $x_1 = \pm 1/\sqrt{3}$.

b) Answer: $x_1 = \pm 1/\sqrt{5}$. (This value can be found by experimentation. It can be also be found by iterating the inverse of the Newton method function.)

Here is an explanation: Using the fact that f is odd and that $x_3 = x_1$ suggests that $x_2 = -x_1$. This greatly simplifies the equation.

$$x_{n+1} = x_n - (x_n - x_n^3)/(1 - 3x_n^2) = \frac{-2x_n^3}{1 - 3x_n^2}$$

Therefore we want to find x satisfying

$$-x = \frac{-2x^3}{1-3x^2}$$

This equation is the same as $x(1-3x^2) = 2x^3$, which implies $x = 0$ or $5x^2 = 1$. In other words, $x = \pm 1/\sqrt{5}$. Now one can check that if $x_1 = 1/\sqrt{5}$, then $x_2 = -1/\sqrt{5}$, $x_3 = +1/\sqrt{5}$, etc.

c) Answers: If $x_1 < -1/\sqrt{3}$, then $x_n \to -1$. If $x_1 > 1/\sqrt{3}$, then $x_n \to 1$. If $-1/\sqrt{5} < x_1 < 1/\sqrt{5}$, then $x_n \to 0$. This can be found experimentally, numerically. For a complete analysis and proof one needs the methods of an upper level course like 18.100.

2F-6 a) To simplify this problem to its essence, let $V = \pi$. (We are looking for ratio r/h and this will be the same no matter what value we pick for V.) Thus $r^2 h = 1$ and

$$A = \pi r^2 + 2\pi/r$$

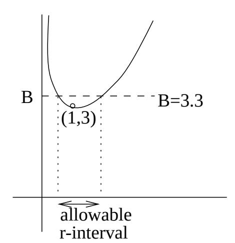
Minimize $B = A/\pi$ instead.

$$B = r^2 + 2r^{-1} \implies B' = 2r - 2r^{-2}$$

and $B' = 0$ implies $r = 1$. Endpoints: $B \to \infty$ as $r \to 0$ and as $r \to \infty$, so we have found the minimum at $r = 1$. (The constraint $r^2 h = 1$ shows that this minimum is achieved when $r = h = 1$. As a doublecheck, the fact that the minimum area is achieved for $r/h = 1$ follows from 2C/5; see part (b).)

The minimum of B is 3 attained at $r = 1$. Ten percent more than the minimum is 3.3, so we need to find all r such that

$$B(r) \le 3.3$$



Use Newton's method with $F(r) = B(r) - 3.3$. (It is unwise to start Newton's method at $r = 1$. Why?) The roots of F are approximately $r = 1.35$ and $r = .72$.

Since $r^2 h = 1$, $h = 1/r^2$ and the ratio,

$$r/h = r^3$$

Compute $(1.35)^3 \approx 2.5$ and $(.72)^3 \approx .37$. Therefore, the proportions with at most 10 percent extra glass are approximately

$$.37 < r/h < 2.5$$

b) The connection with Problem 2C-5 is that the minimum area $r = h$ is not entirely obvious, and not just because we are dealing with glass beakers instead of tin cans. In 2C/5 the area is fixed whereas here the volume is held fixed. But because one needs a larger surface area to hold a larger volume, maximizing volume with fixed area is the same problem as minimizing surface area with fixed volume. This is an important so-called duality principle often used in optimization problems. In Problem 2C-5 the answer was $r = h$, which is the proportion with minimum surface area as confirmed in part (a).

2F-7 Minimize the distance squared, $x^2 + y^2$. The critical points satisfy

$$2x + 2yy' = 0$$

The constraint $y = \cos x$ implies $y' = -\sin x$. Therefore,

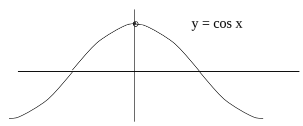
$$0 = x + yy' = x - \cos x \sin x$$

There is one obvious solution $x = 0$. The reason why this problem is in this section is that one needs the tools of inequalities to make sure that there are no other solutions. Indeed, if

$$f(x) = x - \cos x \sin x, \ \text{then } f'(x) = 1 - \cos^2 x + \sin^2 x = 2\sin^2 x \geq 0$$

Furthermore, $f'(x)0$ is strictly positive except at the points $x = k\pi$, so f is increasing and crosses zero exactly once.

There is only one critical point and the distance tends to infinity at the endpoints $x \to \pm\infty$, so this point is the minimum. The point on the graph closest to the origin is $(0, 1)$.



Alternative method: To show that $(0, 1)$ is closest it suffices to show that for $-1 \leq x < 0$ and $0 < x \leq 1$,

$$\sqrt{1 - x^2} < \cos x$$

Squaring gives $1 - x^2 < \cos^2 x$. This can be proved using the principles of problems 6 and 7. The derivative of $\cos^2 x - (1 - x^2)$ is twice the function f above, so the methods are very similar.

2G. Mean-value Theorem

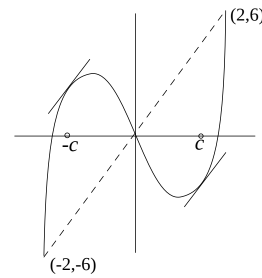
2G-1 a) slope chord = 1; $f'(x) = 2x \Rightarrow f'(c) = 1$ if $c = \frac{1}{2}$.

b) slope chord $= \ln 2$; $f'(x) = \frac{1}{x} \Rightarrow f'(c) = \ln 2$ if $c = \frac{1}{\ln 2}$.

c) for $x^3 - x$: slope chord $= \frac{f(2) - f(-2)}{2-(-2)} = \frac{6-(-6)}{4} = 3$;

$f'(x) = 3x^2 - 1 \Rightarrow f'(c) = 3c^2 - 1 = 3 \Rightarrow c = \pm\frac{2}{\sqrt{3}}$

From the graph, it is clear you should get two values for c. (The axes are not drawn to the same scale.)



2G-2 a) $f(x) = f(a) + f'(c)(x-a)$; Take $a = 0$; $f(\xi) = \sin x$, $f'(x) = \cos x$

$\Rightarrow f(x) = 0 + \cos c \cdot x \quad \Rightarrow \sin x < x$ (since $\cos c < 1$ for $0 < c < 2\pi$)

Thus the inequality is valid for $0 < x \le 2\pi$; since the function is periodic, it is also valid for all $x > 0$.

b) $\frac{d}{dx}\sqrt{1+x} = \frac{1}{2\sqrt{1+x}} \Rightarrow \sqrt{1+x} = 1 + \frac{1}{2\sqrt{1+c}}x < 1 + \frac{1}{2}x$, since $c > 0$.

2G-3. Let $s(t)$ = distance; then average velocity = slope of chord $= \frac{121}{11/6} = 66$.

Therefore, by MVT, there is some time $t = c$ such that $s'(c) = 66 > 65$.

(An application of the mean-value theorem to traffic enforcement...)

2G-4 According to Rolle's Theorem (Thm.1 p.800 : an important special case of the M.V.T, and a step in its proof), between two roots of $p(x)$ lies at least one root of $p'(x)$. Therefore, between the n roots $a_1, \ldots, a_n$ of $p(x)$, lie at least $n-1$ roots of $p'(x)$.

There are no more than $n-1$ roots, since degree of $p'(x) = n-1$; thus $p'(x)$ has exactly $n-1$ roots.

2G-5 Assume $f(x) = 0$ at a, b, c.

By Rolle's theorem (as in MVT-4), there are two points q_1, q_2 where $f'(q_1) = 0, f'(q_2) = 0$.

By Rolle's theorem again, applied to q_1 and q_2 and $f'(x)$, there is a point p where $f''(p) = 0$. Since p is between q_1 and q_2 , it is also between a and c.

2G-6 a) Given two points x_i such that $a \le x_1 < x_2 \le b$, we have

$$f(x_2) = f(x_1) + f'(c)(x_2 - x_1), \quad \text{where } x_1 < c < x_2.$$

Since $f'(x) > 0$ on $[a, b]$, $f'(c) > 0$; also $x_2 - x_1 > 0$. Therefore $f(x_2) > f(x_1)$, which shows $f(x)$ is increasing .

b) We have $f(x) = f(a) + f'(c)(x-a)$ where $a < c < x$.

Since $f'(c) = 0$, $f(x) = f(a)$ for $a \le x \le b$, which shows $f(x)$ is constant on $[a, b]$.