8. Probability

Brief answers to the unstarred exercises are given at the end of this section.

8A. Discrete Random Variables

8A-1 Buck Fuller rolls a fair dodecahedral die: it has 12 faces, all regular pentagons. The outcome is a random variable X, with integer values $1, 2, \ldots, 12$. Find

(a) P(X is divisible by 3) (b) P(X is divisible by 5)

8A-2 A fair pair of dice is rolled; the output is a random variable Y, with values $2, \ldots, 12$. Answer the same two questions as in the preceding exercise.

8A-3 Referring to Example 1.1B: you pick one of the football team's shoes at random. Find the probability of getting an even size; deduce what the probability of getting an odd size is.

8A-4 Assume that when asked to pick a random positive integer, a person picks the integer n with probability $1/2^n$. Let X be the associated random variable giving this outcome. Find

(a) P(X is even); (b) P(X is odd).

(c) Show that $6/16 \le \mathrm{P}(X \text{ is prime}) \le 7/16$.

8A-5 Say Mrs. Field's chocolate chip cookies average 10 chips per cookie. What's the probability of getting 5 or less chips in a cookie? (Assume the number of chips is a Poisson random variable.)

8A-6 Assume the number of calls per night to def-tuv-tuv-oper-oper is a Poisson random variable with mean 5. What's the likelihood that there will be at least three calls tonight?

8A-7 Tabitha is Latexing her thesis, proof-reading as she word-processes. When printed, about 20% of the pages turn out to be error-free. What is the likelihood that

a) a single page has at most one error?

b) three pages have a total of at most three errors?

8A-8 Suppose a calculus textbook has a total of 600 misprints in its 950 pages. What is the probability that

a) a chapter containing 10 pages has no misprints?

b) a chapter 5 pages long has at least one misprint?

8B. Continuous random variables.

8B-1 Let X be an exponential random variable with parameter $m = 2$.

a) Calculate the expectation of X directly from its definition.

b) Calculate $P(1 \le X \le 3)$.

8B-2 The average time between sales at the Chinese pastry booth in the lobby of Building 10 is 4/5 minute (they wish). If we assume the time is an exponential random variable, what is the probability that the time between successive sales is

a) greater than 2 minutes? b) less than 4 minutes?

8B-3 Say that on the average, a baby is born somewhere in the U.S. every 10 seconds (we're assuming it's not 9 months after a massive nighttime power outage). What's the probability of a time gap between two successive births lasting between one and two minutes?

8B-4 Assume the mean length of time between auto accidents on Southeast Expressway is 10 hours. Estimate the probability of no accidents for 24 hours.

8B-5 My city-tire bicycle seems to get a flat on the Charles River bike path on the average every 100 days. For what length of time t_0 will the probability be 90% that I won't get a flat during any time interval of that length?

8B-6 If X is an exponential random variable with parameter m, what is the probability that X exceeds its mean?

8B-7* In Example 2.1,
a) verify the formulas given for the density function and $P(a \leq x \leq b)$;
b) find the distribution function.

3. Standard deviation

8C-1* Find the standard deviation of X if:
a) X is the outcome of tossing a fair die
b) X is the uniform continuous random variable with range $[x_1, x_2]$.

8C-2* In Theorem 3.1, prove: (a) (17) (b) (18)

8C-3* Prove the equality of the two integral formulas in (16) for the variance of a continuous random variable.

4. Normal random variables

8D-1 Let Z be the standard normal random variable. Using the table for the values of the associated distribution function $\Phi(Z)$, extended by (23) and (24), calculate the value of:
a) $P(1.5 < Z < 2.5)$ b) $P(Z \leq -1)$ c) $P(-1 < Z < 1)$ d) $P(-1 < Z < 2.5)$

8D-2 Assume the lifetime in hours of a flashlight battery is a normal random variable X with mean 120 and standard deviation 36. Find the probability that it lasts between 85 and 135 hours. How many batteries in a sample of 160 would you expect to last that long, on the average?

8D-3 Suppose the grades on an .01A test have a normal distribution with mean 70 and standard deviation 10. If 300 students take the test and passing is set at 55, how many fail? A mean professor decides that "keeping up the standards" requires that 10% of the students fail. What will she announce as the passing grade?

8D-4 For each of the following normal random variables, give an interval in which the variable lies with probability 95%.
a) Lifetime in hours of a flashlight battery if $m = 120,\ \sigma = 36$;
b) grade on an exam for which $m = 70,\ \sigma = 10$;
c) annual snowfall in inches, if the mean is $46''$ and the standard deviation $4''$.

8D-5* Prove in Theorem 4.1 that $\sigma(Z) = 1$ for the standard normal random variable Z.

8D-6* Prove the first implication in (22) by making the change of variable.

8D-7* Prove the total area under the normal density function (28) is 1 by making a change of variable in the integral; you can use the results in (20).

8E. Central Limit Theorem

8E-1 Suppose the average luggage weight for an airline passenger is 38 lbs. with a standard deviation of 8 lbs. What is the probability that the luggage for 80 passengers will weigh over 3200 pounds?

8E-2 In an R/O week contest, one hundred freshmen independently estimate the height in meters of a picket fence. Assume that the standard deviation for the individual guesses is less than 1 dm (.1 meter). Give a lower bound on the probability that the average of their guesses is off by less than 1 cm.

8E-3 A national poll is to estimate the percentage of Americans who favor a draft over a volunteer military . Copy and complete the table below so that if n people are polled at random, we can say with approximately 95% confidence that our error is less than e percentage points.

n:	50	100		625		10,000
e:		5	4	3	2	

8E-4 The Today Show announces that in a poll of 900 randomly chosen Americans, 52% favored college tuition tax credits. In what range can you say with approximately 95% confidence that the actual percentage lies?

8E-5 To prove that a coin is unfair, a judge tosses it 2,000 times. How many heads would he need to get to prove with 95% confidence that the coin is unfair?

8E-6 One hundred reservations have been confirmed for the 98-seat flight from Boston to Bangor. If generally 3% of the confirmed passengers do not show up, what is the probability that someone will be bumped from the flight?

8E-7 A poll of 10,000 Bostonians a week before a gubernatorial election gives the incumbent 52% of the vote. In what range can you put his support with approximately 95% confidence?

Answers

8A-1 a) 1/3 b) 1/6
8A-2 a) 1/3 b) 7/36
8A-3 a) 11/24 b) 13/24
8A-4 a) 1/3 b) 2/3 c) P(X is 2 or 3)=6/16; P(X is neither 1 nor 4) = 7/16
8A-5 $e^{-10}(1+10+10^2/2!+\ldots+10^5/5!) = .067$
8A-6 $e^{-5}(1+5+5^2/2!) = .125 = P(X \le 2)$. So $P(X \ge 3) = .875$
8A-7 The mean is .2, so 1.61 errors/page
a) $e^{-1.61}(1+1.61) = .522 =$P(at most 1 error/page)
b) average no. errors in 3 pages is 4.83; P(3 or less)= .29
8A-8 a) .0018 b) .957

8B-1 $E(X) = 2$; $P(1 < X < 3) = e^{-1/2} - e^{-3/2} = 38\%$.
8B-2 $e^{-5/2} \approx 8\%$; $1 - e^{-5} \approx 99\%$.
8B-3 $e^{-6} - e^{-12} \approx 0.2\%$.
8B-4 $e^{-2.4} \approx 9\%$.
8B-5 $e^{-t_0/100} = .9 \;\Rightarrow\; t_0 = -100 \ln 9 \approx 10.5$ days.
8B-6 $e^{-1} \approx 37\%$.

8D-1 $.9938 - .9332 = .0608$ $1 - .8413 = .1587$;
$.8413 - (1 - .8413) = .6826$; $.9938 - (1 - .8413) = .8351$
8D-2 $P(85 < X < 135) = P(-35/36 < Z < 15/36) = .6628 - .1660 = .4968$.
Ans: about 50, about 80.
8D-3 $P(X \le 55) = P(Z \le -15/10) = .0668$. About 20 fail.
$10\% \approx P(Z \ge 1.3) = P(Z \le -1.3) = P(X < 57)$.
8D-4 48 to 192 hrs.; 50 to 90; 38 to 54 inches

8E-1 $\bar{\sigma} = 8/\sqrt{80} = 2/\sqrt{5}$, $P(\bar{X} \ge 40) = P(Z \ge \sqrt{5} \approx 1 - P(Z \le 2.24) = 1 - .987 = .013$, so about 1.3%.
8E-2 $\bar{\sigma} \le .1/\sqrt{100} = .01; P(|\bar{X} - \bar{\sigma}| \le .01) = P(|Z| \le 1) = 2P(0 \le Z \le 1) = 2(P(Z \le 1) - 1/2) = 2(.84 - .50) = .68$; about 68%
8E-3 $e = 100/\sqrt{n} \approx 14, 10, 1$; $n = 10,000/e^2 \approx 400, 1111$.
8E-4 about 49%-55%
8E-5 Less than about 955 or more than about 1045 heads would show coin unfair.
8E-6 $4e^{-3} \approx 20\%$
8E-7 51%-53%