7. Infinite Series

7A. Basic Definitions

7A-1 Do the following series converge or diverge? Give reason. If the series converges, find its sum.

a) $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots + \frac{1}{4^n} + \ldots$ b) $1 - 1 + 1 - 1 + \ldots + (-1)^n + \ldots$

c) $1 + \frac{1}{2} + \frac{2}{3} + \ldots + \frac{n}{n+1} + \ldots$ d) $\ln 2 + \ln \sqrt{2} + \ln \sqrt[3]{2} + \ln \sqrt[4]{2} + \ldots$

e) $\sum_1^\infty \frac{2^{n-1}}{3^n}$ f) $\sum_0^\infty (-1)^n \frac{1}{3^n}$

7A-2 Find the rational number represented by the infinite decimal .21111 . . . .

7A-3 For which x does the series $\sum_0^\infty \left(\frac{x}{2}\right)^n$ converge? For these values, find its sum $f(x)$.

7A-4 Find the sum of these series by first finding the partial sum S_n.

a) $\sum_1^\infty \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

b) $\sum_1^\infty \frac{1}{n(n+2)}$. (Hint: $\frac{1}{n(n+2)} = \frac{a}{n} + \frac{b}{n+2}$ for suitable a, b).

7A-5 A ball is dropped from height h; each time it lands, it bounces back 2/3 of the height from which it previously fell. What is the total distance (up and down) the ball travels?

7B: Convergence Tests

7B-1 Using the integral test, tell whether the following series converge or diverge; show work or reasoning.

a) $\sum_0^\infty \frac{n}{n^2+4}$ b) $\sum_0^\infty \frac{1}{n^2+1}$ c) $\sum_0^\infty \frac{1}{\sqrt{n+1}}$

d) $\sum_1^\infty \frac{\ln n}{n}$ e) $\sum_2^\infty \frac{1}{(\ln n)^p \cdot n}$ f) $\sum_1^\infty \frac{1}{n^p}$

(In the last two, the answer depends on the value of the parameter p.)

7B-2 Using the limit comparison test, tell whether each series converges or diverges; show work or reasoning. (For some of them, simple comparison works.)

a) $\sum_1^\infty \frac{1}{n^2+3n}$ b) $\sum_1^\infty \frac{1}{n+\sqrt{n}}$ c) $\sum_1^\infty \frac{1}{\sqrt{n^2+n}}$

d) $\sum_1^\infty \sin\left(\frac{1}{n^2}\right)$ e) $\sum_1^\infty \frac{\sqrt{n}}{n^2+1}$ f) $\sum_1^\infty \frac{\ln n}{n}$

g) $\sum_2^\infty \frac{n^2}{n^4-1}$ h) $\sum_1^\infty \frac{n^3}{4n^4+n^2}$

7B-3 Prove that if $a_n > 0$ and $\sum_0^\infty a_n$ converges, then $\sum_0^\infty \sin a_n$ also converges.

7B-4 Using the ratio test, or otherwise, determine whether or not each of these series is absolutely convergent. (Note that $0! = 1$.)

a) $\displaystyle\sum_0^\infty \frac{n}{2^n}$ b) $\displaystyle\sum_0^\infty \frac{2^n}{n!}$ c) $\displaystyle\sum_1^\infty \frac{2^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

d) $\displaystyle\sum_0^\infty \frac{(n!)^2}{(2n)!}$ e) $\displaystyle\sum_1^\infty \frac{(-1)^n}{\sqrt{n}}$ f) $\displaystyle\sum_1^\infty \frac{n!}{n^n}$; use $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

g) $\displaystyle\sum_1^\infty \frac{(-1)^n}{n^2}$ h) $\displaystyle\sum_0^\infty \frac{(-1)^n}{\sqrt{n^2+1}}$ i) $\displaystyle\sum_0^\infty \frac{n}{n+1}$

7B-5 For those series in **7B-4** which are *not* absolutely convergent, tell whether they are conditionally convergent or divergent.

7B-6 By using the ratio test, determine the radius of convergence of each of the following power series.

a) $\displaystyle\sum_1^\infty \frac{x^n}{n}$ b) $\displaystyle\sum_1^\infty \frac{2^n x^n}{n^2}$ c) $\displaystyle\sum_0^\infty n!x^n$

d) $\displaystyle\sum_0^\infty \frac{(-1)^n x^{2n}}{3^n}$ e) $\displaystyle\sum_0^\infty \frac{(-1)^n x^{2n+1}}{2^n\sqrt{n}}$ f) $\displaystyle\sum_0^\infty \frac{(2n)!x^{2n}}{(n!)^2}$

g) $\displaystyle\sum_2^\infty \frac{x^n}{\ln n}$ h) $\displaystyle\sum_0^\infty \frac{2^{2n}x^n}{n!}$

7C: Taylor Approximations and Power Series

7C-1 Using the general formula for the coefficients a_n, find the Taylor series at 0 for the following functions; do the work systematically, calculating in order the $f^{(n)}$, $f^{(n)}(0)$, and then the a_n.

a) $\cos x$ b) $\ln(1+x)$ c) $\sqrt{1+x}$

7C-2 Calculate $\sin 1$ using the Taylor series up to the term in x^3. Estimate the accuracy using the remainder term. (The calculator value is .84147.) Use the remainder term $R_6(x)$, not $R_5(x)$; why?

7C-3 Using the remainder term, tell for what value of n in the approximation

$$e^x \approx 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

the resulting calculation will give e to 3 decimal places (by convention, this means: within .0005).

7C-4 By using the remainder term, tell whether $\cos x \approx 1 - \dfrac{x^2}{2!}$ will be valid to within .001 over the interval $|x| < .5$.

7C-5 Calculate $\displaystyle\int_0^{.5} e^{-x^2}dx$, using the approximation for e^{-x^2} up to the term in x^4. Estimate the error, using the correct remainder term (cf. **7B-3**), and tell whether the answer will be good to 3 decimal places.

7D: General Power Series

7D-1 Find the power series around $x = 0$ for each of the following functions by using known Taylor series: use substitution, addition, differentiation, integration, or anything else you can think of:

a) e^{-2x} b) $\cos\sqrt{x},\ x \geq 0$ c) $\sin^2 x$ (use an identity)

d) $\dfrac{1}{(1+x)^2}$ e) $\tan^{-1} x$ (differentiate) f) $\ln(1+x)$

g) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

7D-2 By using operations on power series (substitution, addition, integration, differentiation, multiplication), find the power series for the following functions, and determine the radius of convergence. (Where indicated, give just the first 2 or 3 non-zero terms.)

a) $\dfrac{1}{x+9}$ b) e^{-x^2} c) $e^x \cos x$ (3 terms)

d) $\displaystyle\int_0^x \frac{\sin t}{t}\,dt$ e) $\operatorname{erf} x = \displaystyle\int_0^x e^{-t^2/2}\,dt$ f) $\dfrac{1}{x^3-1}$

g) $\cos^2 x$ (differentiate; then use a trigonometric identity)

h) $\dfrac{\sin x}{1-x}$ (3 terms); do it two ways: multiplication, and dividing $\sin x$ series by $1-x$

i) $\tan x$ (2 terms); do it two ways: Taylor series, and division of power series

7D-3 Find the following limits by using linear, quadratic, or cubic approximation (i.e. by using the first few terms of the Taylor series), *not* by using L'Hospital's rule.

a) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$ b) $\displaystyle\lim_{x\to 0} \frac{x-\sin x}{x^3}$

c) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-1-x/2}{\sin^2 x}$ d) $\displaystyle\lim_{u\to 0} \frac{\cos u - 1}{\ln(1+u)-u}$