Problem 1. (12 points)
Say whether the following series converge or diverge.

(a) \( \sum_{n=0}^{\infty} \left( \frac{1 + 2i}{1-i} \right)^n \)
(b) \( \sum_{n=0}^{\infty} i^n \)
(c) \( \sum_{n=0}^{\infty} \left( \frac{1-i}{1+2i} \right)^n \)
(d) \( \sum_{n=0}^{\infty} \frac{n!}{10^n} \)

Problem 2. (8 points)
Find the radius of convergence.

(a) \( f_1(z) = \sum_{n=0}^{\infty} \frac{z^{3n}}{2^n} \)
(b) \( f_2(z) = 1 + 3(z - 1) + 3(z - 1)^2 + (z - 1)^3 \)

Problem 3. (8 points)
Suppose the radius of convergence of \( \sum_{n=0}^{\infty} a_n z^n \) is \( R \). Find the radius of convergence of each of the following.

(a) \( \sum_{n=0}^{\infty} a_n z^{2n} \)
(b) \( \sum_{n=1}^{\infty} n^{-n} a_n z^n \)

Problem 4. (10 points)
(a) Give a function \( f \) that is analytic in the punctured plane \( \mathbb{C} - \{1\} \), has a simple zero at \( z = 0 \) and an essential singularity at \( z = 1 \).
(b) Suppose \( f \) is analytic and has a zero of order \( m \) at \( z_0 \). Show that \( g(z) = f'(z)/f(z) \) has a simple pole at \( z_0 \) with \( \text{Res}(g, z_0) = m \).

Problem 5. (20 points)
(a) What is the order of the pole of \( f_1(z) = \frac{1}{(2\cos(z) - 2 + z^2)^2} \) at \( z = 0 \).
Hint: Work with \( 1/f_1(z) \).
(b) Find the residue of \( f_2(z) = \frac{z^2 + 1}{2z \cos(z)} \) at \( z = 0 \)
(c) Let \( f_3(z) = \frac{e^z}{z(z+1)^3} \). Find all the isolated singularities and compute the residue at each one.
(d) Find the residue at infinity of \( f_4(z) = \frac{1}{1-z} \).
(e) Let \( f_5(z) = \frac{\cos(z)}{\int_0^z f(w) \, dw} \), where \( f(z) \) is analytic and \( f(0) = 1 \). Find the residue at \( z = 0 \).

**Problem 6.** (10 points)
Write the principal part of each function at the isolated singularity. Compute the corresponding residue.

(a) \( f_1(z) = z^3 e^{1/z} \)

(b) \( f_2(z) = \frac{1 - \cosh(z)}{z^3} \)

**Problem 7.** (8 points)
(a) Let \( f(z) = (1 + z)^a \), computed using the principal branch of \( \log \). Give the Taylor series around \( 0 \).

(b) Does the principal branch of \( \sqrt{z} \) have a Laurent expansion in the domain \( 0 < |z|? \)

**Problem 8.** (15 points)
Using variations of the geometric series find the following series expansions of \( f(z) = \frac{1}{4 - z^2} \) about \( z_0 = 1 \).

(a) The Taylor series. What is the radius of convergence?

(b) The Laurent series on \( 1 < |z - 1| < R_1 \). What is \( R_1 \)

(c) The Laurent series for \( |z - 1| > 3 \).

**Problem 9.** (15 points)
(a) Use the residue theorem to compute \( \int_{|z|=3} \frac{e^{iz}}{z^2(z - 2)(z + 5i)} \, dz \).

(b) Evaluate \( \int_{|z|=1} e^{1/z} \sin(1/z) \, dz \).

(c) Explain why Cauchy’s integral formula can be viewed as a special case of the residue theorem.

**Problem 10.** (15 points)
In this problem we will compute \( \sum_{n=-\infty}^{\infty} \frac{1}{n^2} \) using the residue theorem. The techniques learned here are general. In particular, the use of \( \cot(\pi z) \) is fairly common.

(a) Let \( \phi(z) = \pi \cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} \). At all the singular points give the order the pole and the residue.

(b) Take the contour \( C_N \) which is the square with vertices at \( \pm(N + 1/2) \pm i(N + 1/2) \) Use the Cauchy residue theorem to write an expression for
\[
\int_{C_N} \frac{\pi \cot(\pi z)}{z^2} \, dz
\]
You’ll need to do some work to compute the residue at $z = 0$.

(c) We’ll tell you that $|\cot(\pi z)| < 2$ along the contour $C_N$. Use this to show that

$$\lim_{N \to \infty} \int_{C_N} \frac{\pi \cot(\pi z)}{z^2} \, dz = 0.$$  

(d) Use parts (b) and (c) to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Problems below here are not assigned. Do them just for fun.

**Problem Fun 1.** (No points)

By considering the 3 series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$, $\sum_{n=1}^{\infty} \frac{z^n}{n}$, $\sum_{n=1}^{\infty} z^n$, show that a power series may converge on all, some or no points on the boundary of its disk of convergence.

**Problem Fun 2.** (No points)

Suppose that there exists a function $f(z)$ which is analytic at $z = 0$ and which satisfies the differential equation

$$(1 + z) f'(z) = 2 f(z), \text{ with } f(0) = 1$$

(a) Solve this equation to get a closed-form expression for $f(z)$.

(b) Find the formula for the power series coefficients of $f(z)$ directly from the differential equation.

(c) Check your answer to part (b) against the Taylor series obtained by expanding out the closed-form expression for the solution found in part (a).

**Problem Fun 3.** (No points) Show that $|\cot(\pi z)| < 2$ along the contour in problem 10. Hint, show that along the vertical sides $|\cot(\pi z)| < 1$, while along the horizontal sides $|\cot(\pi z)| < 2$.

**Problem Fun 4.** (No points) Suppose the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is $R$. Show that the radius of convergence of $\sum_{n=0}^{\infty} n^2 a_n z^n$ is also $R$. 

*End of pset 6*