Problem 1. (12 points)
Say whether the following series converge or diverge.
(a) $\sum_{n=0}^{\infty} \left( \frac{1+2i}{1-i} \right)^n$
(b) $\sum_{n=0}^{\infty} i^n$
(c) $\sum_{n=0}^{\infty} \left( \frac{1-i}{1+2i} \right)^n$
(d) $\sum_{n=0}^{\infty} \frac{n!}{10^n}$

Problem 2. (8 points)
Find the radius of convergence.
(a) $f_1(z) = \sum_{n=0}^{\infty} \frac{z^{3n}}{2^n}$
(b) $f_2(z) = 1 + 3(z-1) + 3(z-1)^2 + (z-1)^3$

Problem 3. (8 points)
Suppose the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is $R$. Find the radius of convergence of each of the following.
(a) $\sum_{n=0}^{\infty} a_n z^{2n}$
(b) $\sum_{n=1}^{\infty} n^{-n} a_n z^n$

Problem 4. (10 points)
(a) Give a function $f$ that is analytic in the punctured plane $(\mathbb{C} - \{1\})$, has a simple zero at $z = 0$ and an essential singularity at $z = 1$.
(b) Suppose $f$ is analytic and has a zero of order $m$ at $z_0$. Show that $g(z) = f'(z)/f(z)$ has a simple pole at $z_0$ with $\text{Res}(g, z_0) = m$.

Problem 5. (20 points)
(a) What is the order of the pole of $f_1(z) = \frac{1}{(2\cos(z) - 2 + z^2)^2}$ at $z = 0$.
Hint: Work with $1/f_1(z)$.
(b) Find the residue of $f_2(z) = \frac{z^2 + 1}{2z \cos(z)}$ at $z = 0$.
(c) Let $f_3(z) = \frac{e^z}{z(z+1)^3}$. Find all the isolated singularities and compute the residue at each one.
(d) Find the residue at infinity of $f_4(z) = \frac{1}{1-z}$.
(e) Let \( f_5(z) = \frac{\cos(z)}{\int_0^z f(w) \, dw} \), where \( f(z) \) is analytic and \( f(0) = 1 \). Find the residue at \( z = 0 \).

**Problem 6.** (10 points)
Write the principal part of each function at the isolated singularity. Compute the corresponding residue.

(a) \( f_1(z) = z^3e^{1/z} \)

(b) \( f_2(z) = \frac{1 - \cosh(z)}{z^3} \)

**Problem 7.** (8 points)
(a) Let \( f(z) = (1+z)^a \), computed using the principal branch of log. Give the Taylor series around 0.

(b) Does the principal branch of \( \sqrt{z} \) have a Laurent expansion in the domain \( 0 < |z| \)?

**Problem 8.** (15 points)
Using variations of the geometric series find the following series expansions of
\[ f(z) = \frac{1}{4-z^2} \]
about \( z_0 = 1 \).

(a) The Taylor series. What is the radius of convergence?

(b) The Laurent series on \( 1 < |z-1| < R_1 \). What is \( R_1 \)?

(c) The Laurent series for \( |z-1| > 3 \).

**Problem 9.** (15 points)
(a) Use the residue theorem to compute \( \int_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)} \, dz \).

(b) Evaluate \( \int_{|z|=1} e^{1/z} \sin(1/z) \, dz \).

(c) Explain why Cauchy’s integral formula can be viewed as a special case of the residue theorem.

**Problem 10.** (15 points)
In this problem we will compute \( \sum_{-\infty}^{\infty} \frac{1}{n^2} \) using the residue theorem. The techniques learned here are general. In particular, the use of \( \cot(\pi z) \) is fairly common.

(a) Let \( \phi(z) = \pi \cot(\pi z) = \pi \frac{\cos(\pi z)}{\sin(\pi z)} \). At all the singular points give the order of the pole and the residue.

(b) Take the contour \( C_N \) which is the square with vertices at \( \pm(N + 1/2) \pm i(N + 1/2) \). Use the Cauchy residue theorem to write an expression for
\[ \int_{C_N} \frac{\pi \cot(\pi z)}{z^2} \, dz \].
You’ll need to do some work to compute the residue at \( z = 0 \).

(c) We’ll tell you that \( |\cot(\pi z)| < 2 \) along the contour \( C_N \). Use this to show that

\[
\lim_{N \to \infty} \int_{C_N} \frac{\pi \cot(\pi z)}{z^2} \, dz = 0.
\]

(d) Use parts (b) and (c) to compute \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

Problems below here are not assigned. Do them just for fun.

**Problem Fun 1.** (No points)

By considering the 3 series \( \sum_{n=1}^{\infty} \frac{z^n}{n^2} \), \( \sum_{n=1}^{\infty} \frac{z^n}{n} \), \( \sum_{n=1}^{\infty} z^n \), show that a power series may converge on all, some or no points on the boundary of its disk of convergence.

**Problem Fun 2.** (No points)

Suppose that there exists a function \( f(z) \) which is analytic at \( z = 0 \) and which satisfies the differential equation

\[
(1 + z)f'(z) = 2f(z), \quad \text{with } f(0) = 1.
\]

(a) Solve this equation to get a closed-form expression for \( f(z) \).

(b) Find the formula for the power series coefficients of \( f(z) \) directly from the differential equation.

(c) Check your answer to part (b) against the Taylor series obtained by expanding out the closed-form expression for the solution found in part (a).

**Problem Fun 3.** (No points) Show that \( |\cot(\pi z)| < 2 \) along the contour in problem 10. Hint, show that along the vertical sides \( |\cot(\pi z)| < 1 \), while along the horizontal sides \( |\cot(\pi z)| < 2 \).

**Problem Fun 4.** (No points) Suppose the radius of convergence of \( \sum_{n=0}^{\infty} a_n z^n \) is \( R \). Show that the radius of convergence of \( \sum_{n=0}^{\infty} n^2 a_n z^n \) is also \( R \).

*End of pset 6*