3 Line integrals and Cauchy’s theorem

3.1 Introduction

The basic theme here is that complex line integrals will mirror much of what we’ve seen for multivariable calculus line integrals. But, just like working with $e^{i\theta}$ is easier than working with sine and cosine, complex line integrals are easier to work with than their multivariable analogs. At the same time they will give deep insight into the workings of these integrals.

3.2 Ingredients

- The complex plane: $z = x + iy$
- The complex differential $dz = dx + idy$
- A curve in the complex plane: $\gamma(t) = x(t) + iy(t)$, defined for $a \leq t \leq b$.
- A complex function: $f(z) = u(x, y) + iv(x, y)$

3.3 Complex line integrals

Note. Line integrals are also called path or contour integrals.

Given the ingredients we define the complex line integral $\int_{\gamma} f(z) \, dz$ by

$$\int_{\gamma} f(z) \, dz = \int_{a}^{b} f(\gamma(t))\gamma'(t) \, dt. \tag{1}$$

You should note that this notation looks just like integrals of a real variable. We don’t need the vectors and dot products of line integrals in $\mathbb{R}^2$. Also, make sure you understand that the product $f(\gamma(t))\gamma'(t)$ is just a product of complex numbers.

An alternative notation uses $dz = dx + idy$ to write

$$\int_{\gamma} f(z) \, dz = \int_{\gamma} (u + iv)(dx + idy) \tag{2}$$

Let’s check that Equations 1 and 2 are the same. Equation 2 is really a multivariable calculus expression, so thinking of $\gamma(t)$ as $(x(t), y(t))$ it becomes

$$\int_{\gamma} f(z) \, dz = \int_{a}^{b} (u(x(t), y(t)) + iv(x(t), y(t))(x'(t) + iy'(t)) \, dt$$

But, $u(x(t), y(t)) + iv(x(t), y(t)) = f(\gamma(t))$ and $x'(t) + iy'(t) = \gamma'(t)$ so the right hand side of this equation is

$$\int_{a}^{b} f(\gamma(t))\gamma'(t) \, dt.$$
That is, it is exactly the same as the expression in Equation 1.

**Example 3.1.** Compute \( \int_\gamma z^2 \, dz \) along the straight line from 0 to 1 + i.

**Answer:** We parametrize the curve as \( \gamma(t) = t(1 + i) \) with \( 0 \leq t \leq 1 \). So \( \gamma'(t) = 1 + i \). The line integral is

\[
\int_\gamma z^2 \, dz = \int_0^1 t^2 (1 + i)^2 (1 + i) \, dt = \frac{2i(1 + i)}{3}.
\]

**Example 3.2.** Compute \( \int_\gamma z \, dz \) along the straight line from 0 to 1 + i.

**Answer:** We can use the same parametrization as in the previous example. So,

\[
\int_\gamma z \, dz = \int_0^1 t(1 - i)(1 + i) \, dt = 1.
\]

**Example 3.3.** Compute \( \int_\gamma z^2 \, dz \) along the unit circle.

**Answer:** We parametrize the unit circle by \( \gamma(\theta) = e^{i\theta} \), where \( 0 \leq \theta \leq 2\pi \). We have \( \gamma'(\theta) = ie^{i\theta} \). So, the integral becomes

\[
\int_\gamma z^2 \, dz = \int_0^{2\pi} e^{2i\theta} e^{i\theta} \, d\theta = \int_0^{2\pi} ie^{3i\theta} \, d\theta = \frac{e^{3i\theta}}{3} \bigg|_0^{2\pi} = 0.
\]

### 3.4 Fundamental theorem for complex line integrals

This is exactly analogous to the fundamental theorem of calculus.

**Theorem.** (Fundamental theorem of complex line integrals) If \( f(z) \) is a complex analytic function and \( \gamma \) is a curve from \( z_0 \) to \( z_1 \) then

\[
\int_\gamma f'(z) \, dz = f(z_1) - f(z_0).
\]

**Proof.** This is an application of the chain rule. We have

\[
\frac{df(\gamma(t))}{dt} = f'(\gamma(t)) \gamma'(t).
\]

So

\[
\int_\gamma f'(z) \, dz = \int_a^b f'(\gamma(t)) \gamma'(t) \, dt = \int_a^b \frac{df(\gamma(t))}{dt} \, dt = f(\gamma(t)) \bigg|_a^b = f(z_1) - f(z_0).
\]

Another way to state the fundamental theorem is: if \( f \) has an antiderivative \( F \), i.e. \( F' = f \) then

\[
\int_\gamma f(z) \, dz = F(z_1) - F(z_0).
\]
Example 3.4. Redo $\int_\gamma z^2 \, dz$, with $\gamma$ the straight line from 0 to $1 + i$.

**answer:** We can check by inspection that $z^2$ has an antiderivative $F(z) = z^3/3$. Therefore the fundamental theorem implies

$$\int_\gamma z^2 \, dz = \left. \frac{z^3}{3} \right|_0^{1+i} = \frac{(1+i)^3}{3} = \frac{2i(1+i)}{3}.$$

Example 3.5. Redo $\int_\gamma z^2 \, dz$, with $\gamma$ the unit circle.

**answer:** Again, since $z^2$ had antiderivative $z^3/3$ we can evaluate the integral by plugging the endpoints of $\gamma$ into the $z^3/3$. Since the endpoints are the same the resulting difference will be 0!

### 3.5 Cauchy-Riemann all the way down

We’ve defined an analytic function as one having a complex derivative. The following theorem shows that if $f$ is analytic then so is $f'$. Thus, there are derivatives all the way down!

**Theorem.** If $f(z)$ is analytic then so is $f'(z)$.

**Proof.** To show this we have to prove that $f'(z)$ satisfies the Cauchy-Riemann equations. If $f = u + iv$ we know

$$u_x = v_y; \quad u_y = -v_x; \quad f' = u_x + iv_x.$$

Let’s write $f' = U + iV$, so $U = u_x = v_y$ and $V = v_x = -u_y$. We want to show that $U_x = V_y$ and $U_y = -V_x$. We do them one at a time.

$U_x = V_y:$ $U_x = v_y$ and $V_y = v_{xy}$. Since $v_{xy} = v_{yx}$, we have $U_x = V_y$.

$U_y = -V_x:$ $U_y = u_{xy}$ and $V_x = -u_{yx}$. So, $U_y = -V_x$. QED.

**Technical point.** We’ve assumed as many partials as we need. So far we can’t guarantee that all the partials exist. Soon we will have a theorem which says that an analytic function has derivatives of all order. We’ll just assume that for now. In any case, in most examples this will be obvious.

### 3.6 Path independence

We say the integral $\int_\gamma f(z) \, dz$ is path independent if it has the same value for any two paths with the same endpoints.

More precisely, if $f(z)$ is defined on a region $A$ then $\int_\gamma f(z) \, dz$ is path independent in $D$, if it has the same value for any two paths in $D$ with the same endpoints.

**Theorem.** If $f(z)$ has an antiderivative then the path integral $\int_\gamma f(z) \, dz$ is path independent.

**Proof.** This is the same argument as in Example 3.5. Since $f(z)$ has an antiderivative of $f(z)$, the fundamental theorem tells us that the integral only depends on the endpoint of
γ, i.e.
\[ \int_{\gamma} f(z) \, dz = F(z_1) - F(z_0) \]
where \( z_0 \) and \( z_1 \) are the beginning and end point of \( \gamma \).

An alternative way to express path independence uses closed paths.

**Theorem 3.6.** The following two things are equivalent.

1. The integral \( \int_{\gamma} f(z) \, dz \) is path independent.
2. The integral \( \int_{\gamma} f(z) \, dz \) around any closed path is 0.

**Proof.** This is essentially identical to the equivalent multivariable proof. We have to show two things:

(i) Path independence implies the line integral around any closed path is 0.

(ii) If the line integral around all closed paths is 0 then we have path independence.

To see (i), assume path independence and consider the closed path \( C \) shown in figure (i) below. Since the starting point \( z_0 \) is the same as the endpoint \( z_1 \) the line integral \( \int_{C} f(z) \, dz \) must have the same value as the line integral over the curve consisting of the single point \( z_0 \). Since that is clearly 0 we must have the integral over \( C \) is 0.

To see (ii), assume \( \int_{C} f(z) \, dz = 0 \) for any closed curve. Consider the two curves \( C_1 \) and \( C_2 \) shown in figure (ii). Both start at \( z_0 \) and end at \( z_1 \). By the assumption that integrals over closed paths are 0 we have \( \int_{C_1 - C_2} f(z) \, dz = 0 \). So,

\[ \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz. \]

That is, any two paths from \( z_0 \) to \( z_1 \) have the same line integral. This shows that the line integrals are path independent.

![Figure (i)](image1)

![Figure (ii)](image2)

### 3.7 Examples

**Example 3.7.** Why can’t we compute \( \int_{\gamma} \bar{z} \, dz \) using the fundamental theorem.
answer: Because $\overline{z}$ doesn’t have an antiderivative.

Example 3.8. Compute $\int_\gamma 1/z \, dz$ over several contours

(i) The line from 1 to $1 + i$.
(ii) The circle of radius 1 around $z = 3$.
(iii) The unit circle.

answer: For parts (i) and (ii) there is no problem using the antiderivative $\log(z)$ because these curves are contained in a simply connected region that doesn’t contain origin.

(i) answer: $\log(1 + i) - \log(1) = \log(\sqrt{2}) + i\pi/4$.
(ii) answer: 0 (since the beginning and end points are the same).
(iii) answer: We parametrize the unit circle by $\gamma(\theta) = e^{i\theta}$ with $0 \leq \theta \leq 2\pi$. We compute $\gamma'(\theta) = ie^{i\theta}$. So the integral becomes

$$\int_{\gamma} \frac{1}{z} \, dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} \, dt = \int_0^{2\pi} i \, dt = 2\pi i.$$

Notice that we could use $\log(z)$ if we were careful to let it the argument increase by $2\pi$ as it went around the origin once.

Example 3.9. Compute $\int_\gamma 1/z^2 \, dz$ around unit circle in two ways.

(i) Using the fundamental theorem
(ii) Directly from the definition.

answer: 0. (You should supply the details for this.)

3.8 Cauchy’s theorem

Cauchy’s theorem is analogous to Green’s theorem for curl free vector fields.

Theorem. (Cauchy’s theorem) Suppose $A$ is a simply connected region, $f(z)$ is analytic on $A$ and $C$ is a simple closed curve in $A$. Then the following three things hold:

(i) $\int_C f(z) \, dz = 0$

(i') We can drop the requirement that $C$ is simple in part (i).

(ii) Integrals of $f$ on paths within $A$ are path independent. That is, two paths with same endpoints integrate to the same value.

(iii) $f$ has an antiderivative in $A$.

Proof. We will prove (i) using Green’s theorem, We could give a proof that didn’t rely on Green’s, but it would be quite similar in flavor to the proof of Green’s theorem.

Let $R$ be the region inside the curve. And write $f = u + iv$. Now we write out the integral as follows

$$\int_C f(z) \, dz = \int_C (u + iv) \, (dx + idy) = \int_C (u \, dx - v \, dy) + i(v \, dx + u \, dy).$$

Let’s apply Green’s theorem to the real and imaginary pieces separately. First the real
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\[ \int_C u \, dx - v \, dy = \int_R -v_x - u_y \, dx \, dy = 0. \]

We get 0 because the Cauchy-Riemann equations say \( u_y = -v_x \), so \( -v_x - u_y = 0 \).

Likewise for the imaginary piece:

\[ \int_C v \, dx + u \, dy = \int_R u_x - v_y \, dx \, dy = 0. \]

We get 0 because the Cauchy-Riemann equations say \( u_x = v_y \), so \( u_x - v_y = 0 \).

To see part (i) you should draw a few curves that intersect themselves and convince yourself that they can be broken into a sum of simple closed curves. Thus, (i) follows from (i).

Note: In order to truly prove part (i) we would need a more technically precise definition of simply connected so we could say that all closed curves within \( A \) can be continuously deformed to each other.

Part (ii) follows from (i) and Theorem 3.6

To see (iii), pick a base point \( z_0 \in A \) and let

\[ F(z) = \int_{z_0}^z f(w) \, dw. \]

Here the integral is over any path in \( A \) connecting \( z_0 \) to \( z \). By part (ii), \( F(z) \) is well defined. If we can show that \( F'(z) = f(z) \) then we’ll be done. Doing this amounts to managing the notation to apply the fundamental theorem of calculus and the Cauchy-Riemann equations.

Let’s write \( f(z) = u(x, y) + iv(x, y) \) and \( F(z) = U(x, y) + iV(x, y) \). Then we can write

\[ \frac{\partial f}{\partial x} = u_x + iv_x, \text{ etc.} \]

We can formulate the Cauchy-Riemann equations for \( F(z) \) as

\[ F'(z) = \frac{\partial F}{\partial x} = \frac{1}{i} \frac{\partial F}{\partial y}, \text{ i.e. } F'(z) = U_x + iV_x = \frac{1}{i}(U_y + iV_y) = V_y - iU_y. \quad (3) \]

For reference we note that using the path \( \gamma(t) = x(t) + iy(t) \), with \( \gamma(0) = z_0 \) and \( \gamma(b) = z \) we have

\[ F(z) = \int_{z_0}^z f(z) \, dz = \int_{z_0}^z (u(x,y) + iv(x,y))(dx + idy) = \int_0^b (u(x(t), y(t)) + iv(x(t), y(t))(x'(t) + iy'(t)) \, dt. \quad (4) \]

Our goal now is to prove that the Cauchy-Riemann equations given in Equation 4 hold for \( F(z) \). The figure below shows an arbitrary path from \( z_0 \) to \( z \), which can be used to compute \( F(z) \). To compute the partials of \( F \) we’ll need the straight lines that continue \( C \) to \( z + h \) or \( z + ih \).
First we’ll look at $\frac{\partial F}{\partial x}$. So, fix $z = x + iy$. Looking at the paths in the figure above we have

$$F(z + h) - F(z) = \int_{C+C_x} f(w) \, dw - \int_C f(w) \, dw = \int_{C_x} f(w) \, dw.$$  

The curve $C_x$ is parametrized by $\gamma(t) = x + t + iy$, with $0 \leq t \leq h$. So,

$$\frac{\partial F}{\partial x} = \lim_{h \to 0} \frac{F(z + h) - F(z)}{h} = \lim_{h \to 0} \frac{\int_{C_x} f(w) \, dw}{h} = \lim_{h \to 0} \frac{\int_0^h u(x + t, y) + iv(x + t, y) \, dt}{h} = u(x, y) + iv(x, y) = f(z). \quad (5)$$  

(The second to last equality uses the fundamental theorem of calculus.)

Similarly, we get

$$\frac{1}{i} \frac{\partial F}{\partial y} = \lim_{h \to 0} \frac{F(z + ih) - F(z)}{ih} = \lim_{h \to 0} \frac{\int_{C_y} f(w) \, dw}{ih} = \lim_{h \to 0} \frac{\int_0^h u(x, y + t) + iv(x, y + t) \, dt}{h} = u(x, y) + iv(x, y) = f(z). \quad (6)$$

Together Equations 5 and 6 show

$$f(z) = \frac{\partial F}{\partial x} = \frac{1}{i} \frac{\partial F}{\partial y}$$

By Equation 3 we have shown that $F$ is analytic and $F' = f$. QED

### 3.9 Extensions of Cauchy’s theorem

Cauchy’s theorem requires that the function $f(z)$ be analytic on a simply connected region. In cases where it is not, we can extend it in a useful way.

Suppose $R$ is the region between the two simple closed curves $C_1$ and $C_2$. 

Theorem. (Extended Cauchy’s theorem) If \( f(z) \) is analytic on \( R \) then

\[
\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz
\]

Proof. The proof is based on the following figure. We ‘cut’ both \( C_1 \) and \( C_2 \) and connect them by two copies of \( C_3 \), one in each direction. (In the figure we have drawn the two copies of \( C_3 \) as separate curves, in reality they are the same curve traversed in opposite directions.)

With \( C_3 \) acting as a cut, the region enclosed by \( C_1 + C_3 - C_2 - C_3 \) is simply connected, so Cauchy’s theorem applies. We get

\[
\int_{C_1 + C_3 - C_2 - C_3} f(z) \, dz = 0
\]

Breaking the integral up we have

\[
\int_{C_1} f(z) \, dz + \int_{C_3} f(z) \, dz - \int_{C_2} f(z) \, dz - \int_{C_3} f(z) \, dz = 0.
\]

This clearly implies \( \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \) as claimed.

Example 3.10. Let \( f(z) = 1/z \). \( f(z) \) is clearly defined and analytic on the punctured plane.

Punctured plane: \( \mathbb{C} - \{0\} \)

Question: What values can \( \int_C f(z) \, dz \) take for \( C \) a simple closed curve (positively oriented) in the plane?
**answer:** We have two cases (i) \(C_1\) not around 0 (ii) \(C_2\) around 0

In case (i) Cauchy’s theorem applies directly because the interior does not contain the problem point at the origin. Thus,

\[
\int_{C_1} f(z) \, dz = 0.
\]

For case (ii) we will show that \(\int_{C_2} f(z) \, dz = 2\pi i\).

Let \(C_3\) be a small circle of radius \(a\) centered at 0 and entirely inside \(C_2\). By the extended Cauchy theorem we have

\[
\int_{C_2} f(z) \, dz = \int_{C_3} f(z) \, dz.
\]

Thus, \(\oint_{C_2} f \, dr = \oint_{C_3} f \, dr\).

Using the usual parametrization of a circle we can easily compute that the line integral is

\[
\int_{C_3} f(z) \, dz = \int_{0}^{2\pi} i \, dt = 2\pi i. \quad \text{QED.}
\]

**Answer to the question:** The only possible values are 0 and \(2\pi i\).

We can extend this answer in the following way:

If \(C\) is not simple, then the possible values of \(\int_{C} f(z) \, dz\) are \(2\pi ni\), where \(n\) is the number of times \(C\) goes (counterclockwise) around the origin 0.

**Definition.** \(n\) is called the *winding number* of \(C\) around 0. (\(n\) also equals the number of times \(C\) crosses the positive \(x\)-axis, counting +1 for crossing from below and −1 for crossing from above.)
Example 3.11. A further extension: using the same trick of cutting the region by curves to make it simply connected we can show that if \( f \) is analytic in the region \( R \) shown below then

\[
\int_{C_1-C_2-C_3-C_4} f(z) \, dz = 0.
\]