10 Conformal transformations

10.1 Introduction

In this topic we will look at the notion of conformal maps. This is a geometric notion and it will turn out that analytic functions are automatically conformal. Once we have the general notion, we will look at a specific family of conformal maps: linear fractional transformations. We will look at their geometric properties. As an application we will use them to solve the Dirichlet problem for harmonic functions on the unit disk with specified values on the unit circle. At the end we will return to some questions of fluid flow.
Topic 10 Conformal Maps

Definition (Informal) Conformal maps are functions that preserve the angles between curves.

Example If $f$ is analytic we have seen that $f$ maps grid lines to orthogonal curves.

Example Consider multiplication by $c = a e^{i\phi}$.

Ray at angle $\Theta_1$ \hspace{2cm} Ray at angle $\phi + \Theta_1$

Multiplication by $c$ rotates curves through an angle $\phi$. 
Definition (Precise) A function $f$ defined on a region $A$ is conformal if at every $z_0$ in $A$ there is an $a > 0$ and an angle $\phi$ (depending on $z_0$) such that for any curve $\gamma$ through $z_0$, $f$ rotates the tangent vector at $z_0$ by $\phi$ and scales it by $a$. 

Same angle between tangent vectors.
(Also they are both scaled by the same amount.)

- Infinitesimally $f$ looks like multiplication by $c = ae^{i\phi}$.

Theorem (Operational definition of conformal)
If $f$ is analytic on $A$ and $f'(z_0) \neq 0$ then $f$ is conformal and at $z_0$ we have $c = ae^{i\phi} = f'(z_0)$.
proof Suppose we have the curve $\gamma$ and $\gamma(t_0) = z_0$. Then by the chain rule

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0)) \gamma'(t_0) = f'(z_0) \cdot \gamma'(t_0).$$

So $f$ rotates $\gamma'(t_0)$ by $\text{Arg}(f'(z_0))$ and scales it by $|f'(z_0)|$.

Linear approximation: $f(z) \approx f(z_0) + f'(z_0) z$

So $f(\gamma(t)) \approx f(z_0) + f'(z_0) \gamma(t)$

Shift Scale and rotate.

For us Conformal $\iff$ Analytic $f'(z) \neq 0$.

Theorem If $u, v$ are harmonic conjugates and $g = u + iv$ has $g(z_0) \neq 0$ then the level curves of $u$ and $v$ through $z_0$ are orthogonal.

(We knew this already using Cauchy-Riemann.)

(We knew this already using Cauchy-Riemann.)
Since \( g \) maps level curves to grid lines
\[(u(x, y) = c \rightarrow \text{Re}(g(x+iy) = c)\]

\(g\) takes level curves to grid lines)

and grid lines are orthogonal. The conformality of \( g \) implies the level curves are orthogonal.

**Riemann Mapping Theorem**

If a region \( A \) is simply connected and not the whole plane, then there is a bijective conformal map from \( A \) to the unit disk.

**Corollary** Any two such regions are conformally equivalent.

**Proof** Will skip. In practice we will write down explicit conformal maps between regions.
10.2 Fractional linear transformations

Definition. A fractional linear transformation is a function of the form
\[ T(z) = \frac{az + b}{cz + d}, \]
where \( a, b, c, d \) are constants and \( ad - bc \neq 0 \).

These are also called Mobius transforms or bilinear transforms. We will abbreviate fractional linear transformation as FLT.

Note. If \( ad - bc = 0 \) then \( T(z) \) is a constant function.

Proof. The full proof requires that we deal with all the cases where some of the coefficients are 0. We'll give the proof assuming \( c \neq 0 \) and leave the case \( c = 0 \) to you. Assuming \( c \neq 0 \), the condition \( ad - bc = 0 \) implies \( \frac{a}{c} (c, d) = (a, b) \). So,
\[
T(z) = \frac{(a/c)(cz + d)}{cz + d} = \frac{a}{c}.
\]
That is, \( T(z) \) is constant.

Note 2. It will be convenient to consider linear transformations to be defined on the extended complex plane \( \mathbb{C} \cup \{\infty\} \). We define
\[
T(\infty) = \begin{cases} 
\frac{a}{c} & \text{if } c \neq 0 \\
\infty & \text{if } c = 0.
\end{cases}
\text{ and } T(-d/c) = \infty \text{ if } c \neq 0.
\]

10.2.1 Examples

Example 10.1. Scale and rotate. Let \( T(z) = az \). If \( a = r \) is real this scales the plane. If \( a = e^{i\theta} \) it rotates the plane. If \( a = re^{i\theta} \) it does both at once.

You supply a figure showing this rotation.

Example 10.2. Scale and rotate and translate. Let \( T(z) = az + b \). Adding the \( b \) term introduces a translation to the previous example.

You supply a figure showing this LFT.

Example 10.3. Inversion. Let \( T(z) = 1/z \). This is called an inversion. It turns the unit circle inside out. Note that \( T(0) = \infty \) and \( T(\infty) = 0 \)

You supply a figure showing inversion.

Example 10.4. Let \( T(z) = \frac{z - i}{z + i} \). We claim that this maps the \( x \)-axis to the unit circle and the upper half-plane to the unit disk.

You supply a figure showing \( T(z) \).

Proof. First take \( x \) real, then \( |T(x)| = \frac{|x - i|}{|x + i|} = \frac{\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} = 1 \). So, \( T \) maps the \( x \)-axis to the unit circle.

Next take \( z = x + iy \) with \( y > 0 \), i.e. \( z \) in the upper half-plane. Clearly \( |y + 1| > |y - 1| \), so \( |z + i| = |x + i(y + 1)| > |x + i(y - 1)| = |z - i| \), that is \( |T(z)| = \frac{|z - i|}{|z + i|} < 1 \). So, \( T \) maps the upper half-plane to the unit disk.
We will use this map frequently, so for the record we note that
\[ T(i) = 0, \quad T(\infty) = 1, \quad T(-1) = i, \quad T(0) = -1, \quad T(1) = -i. \]
These computations show that the real axis is mapped counterclockwise around the unit circle starting at 1 and coming back to 1.

10.2.2 Lines and circles

**Theorem.** A linear fractional transformation maps lines and circles to lines and circles.

Before proving this, note that it does not say lines are mapped to lines and circles to circles. For example, in Example 10.4 the real axis is mapped the unit circle. You can also check that inversion \( w = 1/z \) maps the line \( z = 1 + iy \) to the circle \(|z - 1/2| = 1/2\).

**Proof.** We start by showing that inversion maps lines and circles to lines and circles. Given \( z \) and \( w = 1/z \) we define \( x, y, u \) and \( v \) by
\[
z = x + iy \quad \text{and} \quad w = \frac{1}{z} = \frac{x - iy}{x^2 + y^2} = u + iv.
\]
So, \( u = \frac{x}{x^2 + y^2} \) and \( v = -\frac{y}{x^2 + y^2} \). Now, every circle or line can be described by the equation
\[
Ax + By + C(x^2 + y^2) = D
\]
(If \( C = 0 \) it describes a line, otherwise a circle.) We convert this to an equation in \( u, v \) as follows.
\[
Ax + By + C(x^2 + y^2) = D \iff \frac{Ax}{x^2 + y^2} + \frac{By}{x^2 + y^2} + C = \frac{D}{x^2 + y^2} \iff Au - Bv + C = D(u^2 + v^2).
\]
In the last step we used the fact that \( u^2 + v^2 = |w|^2 = 1/|z|^2 = 1/(x^2 + y^2) \). We have shown that a line or circle in \( x, y \) is transformed to a line or circle in \( u, v \). This shows that inversion maps lines and circles to lines and circles.

We note that for the inversion \( w = 1/z \).

1. Any line not through the origin is mapped to a circle through the origin.
2. Any line through the origin is mapped to a line through the origin.
3. Any circle not through the origin is mapped to a circle not through the origin.
4. Any circle through the origin is mapped to a line not through the origin.

Now, to prove that an arbitrary linear fractional transformation maps lines and circles to lines and circles, we factor it into a sequence of simpler transformations.

First suppose that \( c = 0 \). So, \( T(z) = (az + b)/d \). Since this is just translation, scaling and rotating, it is clear it maps circles to circles and lines to lines.

Now suppose that \( c \neq 0 \). Then,
\[
T(z) = \frac{az + b}{cz + d} = \frac{\frac{a}{c}(cz + d) + b - \frac{ad}{c}}{cz + d} = \frac{a}{c} + \frac{b - ad/c}{cz + d}.
\]
So, \( w = T(z) \) can be computed as a composition of transforms
\[
z \to w_1 = cz + d \to w_2 = 1/w_1 \to w = \frac{a}{c} + (b - ad/c)w_2
\]
We know that each of the transforms in this sequence maps lines and circles to lines and circles. Therefore the entire sequence does also. QED
10.2.3 Mapping $z_j$ to $w_j$

It turns out that for two sets of three points $z_1, z_2, z_3$ and $w_1, w_2, w_3$ there is a linear fractional transformation that takes $z_j$ to $w_j$. We can construct this map as follows.

Let $T_1(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$. Notice that $T_1(z_1) = 0$, $T_1(z_2) = 1$, $T_1(z_3) = \infty$.

Likewise let $T_2(w) = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$. Notice that $T_2(w_1) = 0$, $T_2(w_2) = 1$, $T_2(w_3) = \infty$.

Now $T(z) = T_2^{-1} \circ T_1(z)$ is the required map.

10.2.4 Correspondence with matrices

We can identify the transformation $T(z) = \frac{az + b}{cz + d}$ with the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This identification is useful because of the following algebraic facts.

1. If $r \neq 0$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $r \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ correspond to the same FLT.

**Proof.** This follows from the obvious equality $\frac{az + b}{cz + d} = \frac{raz + rb}{rcz + rd}$.

2. If $T(z)$ corresponds to $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $S(z)$ corresponds to $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ then composition $T \circ S(z)$ corresponds to matrix multiplication $AB$.

**Proof.** The proof is just a bit of algebra.

$$T \circ S(z) = T \left( \frac{ez + f}{gz + h} \right) = \frac{a((ez + f)/(gz + h)) + b}{c((ez + f)/(gz + h)) + d} = \frac{(ae + bg)z + af + bh}{(ce + dg)z + cf + dh}$$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

The claimed correspondence is clear from the last entries in the two lines above.

3. If $T(z)$ corresponds to $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $T$ has an inverse and $T^{-1}(w)$ corresponds to $A^{-1}$ and also to $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, i.e. to $A^{-1}$ without the factor of $1/\det(A)$.

**Proof.** Since $AA^{-1} = I$ it is clear from the previous fact that $T^{-1}$ corresponds to $A^{-1}$. Since

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Fact 1 implies $A^{-1}$ and $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ both correspond to the same FLT, i.e. to $T^{-1}$.

**Example 10.5.**
1. The matrix \[
\begin{bmatrix}
a & b \\
0 & 1 \\
\end{bmatrix}
\] corresponds to \( T(z) = az + b \).

2. The matrix \[
\begin{bmatrix}
e^{i\alpha} & 0 \\
0 & e^{i\alpha} \\
\end{bmatrix}
\] corresponds to rotation by \( 2\alpha \).

3. The matrix \[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\] corresponds to the inversion \( w = 1/z \).

10.3 Reflection and symmetry

10.3.1 Reflection and symmetry in a line

Example 10.6. Suppose we have a line \( S \) and a point \( z_1 \) not on \( S \). The reflection of \( z_1 \) in \( S \) is the point \( z_2 \) so that \( S \) is the perpendicular bisector to the line segment \( \overline{z_1z_2} \). Since there is exactly one such point \( z_2 \), the reflection of a point in a line is unique.

You supply the figure.

Definition. We also say that \( z_1 \) and \( z_2 \) are symmetric with respect to the line \( S \).

In order to define the reflection of a point in a circle we need to work a little harder. Looking back at the previous example we can show the following.

Fact. Any circle through \( z_1 \) and \( z_2 \) intersects \( S \) orthogonally.

Proof. Call the circle \( C \). Since \( S \) is the perpendicular bisector of a chord of \( C \), the center of \( C \) lies on \( S \). Therefore \( S \) is a radial line, i.e. it intersects \( C \) orthogonally.

You supply the figure.

10.3.2 Reflection and symmetry in a circle

We will adapt this for our definition of reflection in a circle. So that the logic flows correctly we need to start with the definition of symmetric pairs of points.

Definition. Suppose \( S \) is a line or circle. A pair of points \( z_1, z_2 \) is called symmetric with respect to \( S \) if every line or circle through the two points intersects \( S \) orthogonally.

First we state an almost trivial fact.

Fact. Linear fractional transformations preserve symmetry. That is, if \( z_1 \) and \( z_2 \) are symmetric in a line or circle \( S \), then, for an LFT \( T \), \( T(z_1) \) and \( T(z_2) \) are symmetric in \( T(S) \).

Proof. The definition of symmetry is in terms of lines and circles, and angles. Linear fractional transformations map lines and circles to lines and circles and, being conformal, preserve angles. QED

Theorem. Suppose \( S \) is a line or circle and \( z_1 \) a point not on \( S \). There is a unique point \( z_2 \) such the pair \( z_1, z_2 \) is symmetric in \( S \).

Proof. Let \( T \) be a linear fractional transformation that maps \( S \) to a line. We know that \( w_1 = T(z_1) \) has a unique reflection \( w_2 \) in this line. Since \( T^{-1} \) preserves symmetry, \( z_1 \) and \( z_2 = T^{-1}(w_2) \) are symmetric in \( S \). Since \( w_2 \) is the unique point symmetric to \( w_1 \) the same is true for \( z_2 \) vis-a-vis \( z_1 \). This is all shown in the figure below.
We can now define reflection in a circle.

**Definition.** The point $z_2$ in the theorem is called the reflection of $z_1$ in $S$.

### 10.3.3 Reflection in the unit circle

Using the symmetry preserving feature of linear fractional transformations, we start with a line and transform to the circle. Let $R$ be the real axis and $C$ the unit circle. We know the LFT $T(z) = \frac{z - i}{z + i}$ maps $R$ to $C$. We also know that the points $z$ and $\overline{z}$ are symmetric in $R$. Therefore

$$w_1 = T(z) = \frac{w - i}{w + i} \quad \text{and} \quad w_2 = T(\overline{z}) = \frac{\overline{z} - i}{\overline{z} + i}$$

are symmetric in $D$. Looking at the formulas, it is clear that $w_2 = 1/w_1$. This is important enough that we highlight it as a theorem.

**Theorem.** Reflection in the unit circle. The reflection of $z$ in the unit circle is $1/\overline{z}$. That is, the reflection of $re^{i\theta}$ is $e^{i\theta}/r$.

**Note.** It is possible, but more tedious and less insightful, to arrive at this theorem by direct calculation.

You supply a figure displaying all this.

**Example 10.7.** Reflection in the circle of radius $R$. Suppose $S$ is the circle $|z| = R$ and $z_1$ is a point not on $S$. Find the reflection of $z_1$ in $S$.

**answer:** Our strategy is to map $S$ to the unit circle, find the reflection and then map the unit circle back to $S$.

You supply a flow diagram illustrating this.

Start with the map $T(z) = w = z/R$. Clearly $T$ maps $S$ to the unit circle and $w_1 = T(z_1) = z_1/R$. The reflection of $w_1$ is $w_2 = 1/\overline{w_1} = R/\overline{z_1}$. Mapping back to the unit circle by $T^{-1}$ we have $z_2 = T^{-1}(w_2) = Rw_2 = R^2/\overline{z_1}$. Therefore the reflection of $z_1$ is $R^2/\overline{z_1}$.

**Example 10.8.** Find the reflection of $z_1$ in the circle of radius $R$ centered at $c$.

**answer:** Let $T(z) = z - c$. $T$ maps the circle centered at $c$ to one centered at 0. So, the reflection of $z_1$ is $z_2 = \frac{1}{z_1 - c} + c$.

We can now record the following important fact.
Fact. Reflection of the center. For a circle $S$ with center $c$, $\infty$ is symmetric with respect to the circle.

Proof. This is an immediate consequence of the formula for the reflection of a point in a circle. For example, the reflection of $z$ in the unit circle is $1/z$. So, the reflection of 0 is infinity.

Example 10.9. Show that if a circle and a line don’t intersect then there is a pair of points $z_1, z_2$ that is symmetric with respect to both the line and circle.

answer: By shifting, scaling and rotating we can find a fractional linear transformation $T$ that maps the circle and line to the following configuration: The circle is mapped to the unit circle and the line to the vertical line $x = a > 1$.

For any real $r$, $w_1 = r$ and $w_2 = 1/r$ are symmetric in the unit circle. We can choose a specific $r$ so that $r$ and $1/r$ are equidistant from $a$, i.e. also symmetric in the line $x = a$. It is clear geometrically that this can be done. Algebraically we solve the equation

$$\frac{r + 1/r}{2} = a \Rightarrow r^2 - 2ar + 1 = 0 \Rightarrow r = a + \sqrt{a^2 - 1}, \quad \frac{1}{r} = a - \sqrt{a^2 - 1}.$$ 

Thus $z_1 = T^{-1}(a + \sqrt{a^2 - 1})$ and $z_2 = T^{-1}(a - \sqrt{a^2 - 1})$ are the required points.

Example 10.10. Show that if two circles don’t intersect then there is a pair of points $z_1, z_2$ that is symmetric with respect to both circles.

answer: Using a linear fractional transformation that maps one of the circles to a line (and the other to a circle) we can reduce the problem to that in the previous example.

Example 10.11. Show that any two circles that don’t intersect can be mapped conformally to concentric circles.

answer: Call the circles $S_1$ and $S_2$. Using the previous example start with a pair of points $z_1, z_2$ which are symmetric in both circles. Next, pick a linear fractional transformation $T$ that maps $z_1$ to 0 and $z_2$ to infinity. For example, $T(z) = \frac{z - z_1}{z - z_2}$. Since $T$ preserves symmetry $0$ and $\infty$ are symmetric in the circle $T(S_1)$. This implies that $0$ is the center of $T(S_1)$. Likewise $0$ is the center of $T(S_2)$. Thus, $T(S_1)$ and $T(S_2)$ are concentric.

10.4 Solving the Dirichlet problem for harmonic functions

In general, a Dirichlet problem in a region $A$ asks you to solve a partial differential equation in $A$ where the values of the solution on the boundary of $A$ are specified.

Example 10.12. Find a function $u$ harmonic on the unit disk such that

$$u(e^{i\theta}) = \begin{cases} 
1 & \text{for } 0 < \theta < \pi \\
0 & \text{for } -\pi < \theta < 0
\end{cases}$$
This is a Dirichlet problem because the values of $u$ on the boundary are specified. The partial differential equation is implied by requiring that $u$ be harmonic, i.e. we require $\nabla^2 u = 0$. We will solve this problem in due course.

### 10.4.1 Harmonic functions on the upper half-plane

Our strategy will be to solve the Dirichlet problem for harmonic functions on the upper half-plane and then transfer these solutions to other domains.

**Example 10.13.** Find a harmonic function $u(x, y)$ on the upper half-plane that satisfies the boundary condition $u(x, 0) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x > 0 \end{cases}$

**answer:** We can write down a solution explicitly as

$$u(x, y) = \frac{1}{\pi} \theta,$$

where $\theta$ is the argument of $z = x + iy$. Since we are only working on the upper half-plane, we can take any convenient branch with branch cut in the lower half-plane, say $-\pi/2 < \theta < 3\pi/2$.

\[\begin{array}{c}
0 \leq y \\
\theta \end{array}\]

To show $u$ is truly a solution, we have to verify two things:

1. $u$ satisfies the boundary conditions
2. $u$ is harmonic.

Both of these are straightforward. First, look at the point $r_2$ on the positive $x$-axis. This has argument $\theta = 0$, so $u(r_2, 0) = 0$. Likewise $\text{arg}(r_1) = \pi$, so $u(r_1, 0) = 1$. Thus, we have shown point (1).

To see point (2) remember that $\log(z) = \log(r) + i\theta$. So, $u = \text{Re}\left(\frac{1}{\pi i} \log(z)\right)$. Since it is the real part of an analytic function, $u$ is harmonic.

**Example 10.14.** Suppose $x_1 < x_2 < x_3$. Find a harmonic function $u$ on the upper half-plane that satisfies the boundary condition $u(x, 0) = \begin{cases} c_0 & \text{for } x < x_1 \\ c_1 & \text{for } x_1 < x < x_2 \\ c_2 & \text{for } x_2 < x < x_3 \\ c_3 & \text{for } x_3 < x \end{cases}$
answer: We mimic the previous example and write down the solution

\[ u(x, y) = c_3 + (c_2 - c_3) \frac{\theta_3}{\pi} + (c_1 - c_2) \frac{\theta_2}{\pi} + (c_1 - c_0) \frac{\theta_1}{\pi}. \]

Here, the \( \theta_j \) are the angles shown in the figure. We again choose the branch \(-\pi/2 < \theta_j < 3\pi/2\).

To convince yourself that \( u \) satisfies the boundary condition test a few points:
At \( r_3 \): all the \( \theta_j = 0 \). So, \( u(r_3, 0) = c_3 \) as required.
At \( r_2 \): \( \theta_1 = \theta_2 = 0, \theta_3 = \pi \). So, \( u(r_2, 0) = c_3 + c_2 - c_3 = c_2 \) as required.
Likewise, at \( r_1 \) and \( r_0 \), \( u \) have the correct values.
As before, \( u \) is harmonic because it is the real part of the analytic function

\[
\Phi(z) = c_3 + \frac{(c_2 - c_3)}{\pi i} \log(z - x_3) + \frac{(c_1 - c_2)}{\pi i} \log(z - x_2) + \frac{(c_1 - c_0)}{\pi i} \log(z - x_1).
\]

10.4.2 Harmonic functions on the unit disk

Let’s try to solve a problem similar to the one in Example 10.12.

Example 10.15. Find a function \( u \) harmonic on the unit disk such that

\[
u(e^{i\theta}) = \begin{cases} 1 & \text{for } -\pi/2 < \theta < \pi/2 \\ 0 & \text{for } \pi/2 < \theta < 3\pi/2 \end{cases}
\]

answer: Our strategy is to start with a conformal map \( T \) from the upper half-plane to the unit disk. We can use this map to pull the problem back to the upper half-plane. We solve it there and then push the solution back to the disk.

Let’s call the disk \( D \), the upper half-plane \( H \). Let \( z \) be the variable on \( D \) and \( w \) the variable on \( H \). Back in Example 10.4 we found a map from \( H \) to \( D \). The map and its inverse are

\[
z = T(w) = \frac{w - i}{w + i}, \quad w = T^{-1}(z) = \frac{iz + i}{-z + 1}.
\]
The function \( u \) on \( D \) is transformed by \( T \) to a function \( \phi \) on \( H \). The relationships are

\[
\phi(z) = \phi \circ T^{-1}(z) \quad \text{or} \quad \phi(w) = u \circ T(w)
\]

These relationships determine the boundary values of \( \phi \) from those we were given for \( u \). We compute:

\[
T^{-1}(i) = -1, \quad T^{-1}(-i) = 1, \quad T^{-1}(1) = \infty, \quad T^{-1}(-1) = 0.
\]

This shows the left hand semicircle bounding \( D \) is mapped to the segment \([-1, 1]\) on the real axis. Likewise, the right hand semicircle maps to the two half-lines shown. (Literally, to the ‘segment’ 1 to \( \infty \) to \(-1\).)

We know how to solve the problem for a harmonic \( \phi \) on \( H \):

\[
\phi(w) = 1 - \frac{1}{\pi} \theta_2 + \frac{1}{\pi} \theta_1 = \text{Re} \left( 1 - \frac{1}{\pi i} \log(w - 1) + \frac{1}{\pi i} \log(w + 1) \right).
\]

Transforming this back to the disk we have

\[
u(z) = \phi \circ T^{-1}(z) = \text{Re} \left( 1 - \frac{1}{\pi i} \log(T^{-1}(z) - 1) + \frac{1}{\pi i} \log(T^{-1}(z) + 1) \right).
\]

If we wanted to, we could simplify this somewhat using the formula for \( T^{-1} \).

### 10.5 Flows around cylinders

#### 10.5.1 Milne-Thomson circle theorem

The Milne-Thomson theorem allows us to insert a circle into a two-dimensional flow and see how the flow adjusts. First we’ll state and prove the theorem.

**Theorem.** **Milne-Thomson circle theorem.** If \( f(z) \) is a complex potential with all its singularities outside \(|z| = R\) then

\[
\Phi(z) = f(z) + f \left( \frac{R^2}{\overline{z}} \right)
\]

is a complex potential with streamline on \(|z| = R\) and the same singularities as \( f \) in the region \(|z| > R\).

**Proof.** First note that \( R^2/\overline{z} \) is the reflection of \( z \) in the circle \(|z| = R\).
Next we need to see that \( f(R^2/z) \) is analytic for \( |z| > R \). By assumption \( f(z) \) is analytic for \( |z| \leq R \), so it can be expressed as a Taylor series

\[
f(z) = a_0 + a_1z + a_2z^2 + \ldots
\]

Therefore,

\[
f\left(\frac{R^2}{z}\right) = a_0 + a_1\frac{R^2}{z} + a_2\left(\frac{R^2}{z}\right)^2 + \ldots
\]

All the singularities of \( f \) are outside \( |z| = R \), so the Taylor series in Equation 1 converges for \( |z| \leq R \). This means the Laurent series in Equation 2 converges for \( |z| \geq R \). That is, \( f(R^2/z) \) is analytic for \( |z| \geq R \), i.e. it introduces no singularities to \( \Phi(z) \) outside \( |z| = R \).

The last thing to show is that \( |z| = R \) is a streamline for \( \Phi(z) \). This follows because for \( z = Re^{i\theta} \),

\[
\Phi(Re^{i\theta}) = f(Re^{i\theta}) + f(Re^{i\theta})
\]

is real. Therefore \( \psi(Re^{i\theta}) = \text{Im}(\Phi(Re^{i\theta})) = 0 \).

### 10.5.2 Examples

Think of \( f(z) \) as representing flow, possibly with sources or vortices outside \( |z| = R \). Then \( \Phi(z) \) represents the new flow when a circular obstacle is placed in the flow. Here are a few examples.

**Example 10.16. Uniform flow around a circle.** We know \( f(z) = z \) is the complex potential for uniform flow to the right. So, \( \Phi(z) = z + R^2/z \) is the potential for uniform flow around a circle of radius \( R \) centered at the origin.

![Uniform flow around a circle](image)

Just because they look nice, the figure includes streamlines inside the circle. These don’t interact with the flow outside the circle.

Note, that as \( z \) gets large flow looks uniform. We can see this analytically because \( \Phi'(z) = 1 - R^2/z^2 \) goes to 0 as \( z \) gets large. (Recall that the velocity yield is \( (\phi_x,-\phi_y) \), where \( \Phi = \phi + i\psi \ldots \))

**Example 10.17. Source flow around a circle.** Here the source is at \( z = -2 \) (outside the unit circle) with complex potential \( f(z) = \log(z+2) \). With the appropriate branch cut the singularities of \( f \) are also outside \( |z| = 1 \). So, we can apply Milne-Thomson and

\[
\Phi(z) = \log(z+2) + \log\left(\frac{1}{z+2}\right)
\]
We know that far from the origin the flow should look the same as a flow with just a source at \( z = -2 \). Let’s see this analytically. First we state a useful fact:

**Useful fact.** If \( g(z) \) is analytic then so is \( h(z) = \overline{g(z)} \) and \( h'(z) = \overline{g'(z)} \).

**Proof.** Use the Taylor series for \( g \) to get the Taylor series for \( h \) and then compare \( h'(z) \) and \( g'(z) \).

Using this we have

\[
\Phi'(z) = \frac{1}{z + 2} - \frac{1}{z(1 + 2z)}
\]

For large \( z \) the second term decays much faster than the first, so \( \Phi'(z) \approx 1/(z + 2) \). That is, the velocity field looks just like the velocity field for \( f(z) \), i.e. the velocity field of a source at \( z = -2 \).

**Example 10.18. Transforming flows.** If we use \( g(z) = z^2 \) we can transform a flow from the upper half-plane to the first quadrant.