Green’s Theorem
Jeremy Orloff

1 Line integrals and Green’s theorem

1.1 Vector Fields (or vector valued functions)

Vector notation. In 18.04 we will mostly use the notation \((v) = (a, b)\) for vectors. The other common notation \((v) = ai + bj\) runs the risk of \(i\) being confused with \(i = \sqrt{-1}\) –especially if I forget to make \(i\) boldfaced.

Definition. A vector field (also called called a vector-valued function) is a function \(F(x, y)\) from \(\mathbb{R}^2\) to \(\mathbb{R}^2\). That is,

\[ F(x, y) = (M(x, y), N(x, y)), \]

where \(M\) and \(N\) are regular functions on the plane. In standard physics notation

\[ F(x, y) = M(x, y)i + N(x, y)j = (M, N). \]

So algebraically, a vector field is nothing more than two ordinary functions of two variables.

Example GT.1. (a.1) Force, constant gravitational field  \(F(x, y) = (0, -g)\).

(a.2) Velocity

\[ V(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right) = \left(\frac{x}{r^2}, \frac{y}{r^2}\right). \]

(Here \(r\) is our usual polar \(r\).) It is a radial vector field, i.e. it points radially away from the origin. It is a shrinking radial field –like water pouring from a source at \((0,0)\).

This vector field exhibits another important feature for us: it is not defined at the origin because the denominator becomes zero there. We will say that \(V\) has a singularity at the origin.

(a.3) Unit tangential field  \(F = (-y, x)/r\). Tangential means tangent to circles centered at the origin. We know it is tangential because it is orthogonal to the radial vector field in (a.2). \(F\) also has a singularity at the origin. We

(a.4) Gradient field  \(F = \nabla f\), e.g., \(f(x, y) = xy^2 \Rightarrow \nabla f = (y^2, 2xy)\).

1.1.1 Visualization of vector fields

This can be summarized as: draw little arrows in the plane. More specifically, for a field \(F\), at each of a number of points \((x, y)\) draw the vector \(F(x, y)\)

Example GT.2. Sketch the vector fields, (a.1), (a.2) and (a.3) from the previous example.
1.2 Definition and computation of line integrals along a parametrized curve

Line integrals are also called path or contour integrals.

We need the following ingredients:
- A vector field \( \mathbf{F}(x, y) = M(x, y) \mathbf{i} + N(x, y) \mathbf{j} = (M, N) \)
- A parametrized curve \( \mathbf{C}: \mathbf{r}(t) = (x(t), y(t)) = (x(t), y(t)), \) with \( t \) running from \( a \) to \( b \).

Note: since \( \mathbf{r} = (x, y) \), we have \( d\mathbf{r} = (dx, dy) \).

**Definition.** The line integral of \( \mathbf{F} \) along \( \mathbf{C} \) is defined as

\[
\oint_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{C}} (M, N) \cdot (dx, dy) = \int_{\mathbf{C}} M \, dx + N \, dy.
\]

Comment: The notation \( \mathbf{F} \cdot d\mathbf{r} \) is common in physics and \( M \, dx + N \, dy \) in thermodynamics. (Though everyone uses both notations.)

We'll see what these notations mean in practice with some examples.

**Example GT.3.** Let \( \mathbf{F}(x, y) = (x^2 y, x - 2y) \) and let \( \mathbf{C} \) be the curve \( \mathbf{r}(t) = (t, t^2) \), with \( t \) running from 0 to 1. Compute the line integral \( I = \oint_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} \).

Do this first using the notation \( \oint_{\mathbf{C}} M \, dx + N \, dy \). Then repeat the computation using the notation \( \oint_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} \).

**answer:** First we draw the curve, which is the part of the parabola \( y = x^2 \) running from \((0,0)\) to \((1,1)\).
(i) Using the notation \( \int_C M \, dx + N \, dy \).

We have \( \mathbf{r} = (x, y) \), so \( x = t, \ y = t^2 \). In this notation \( \mathbf{F} = (M, N) \), so \( M = x^2y \) and \( N = x - 2y \).

We put everything in terms of \( t \):

\[
\begin{align*}
    dx &= dt \\
    dy &= 2t \, dt \\
    M &= (t^2)(t^2) = t^4 \\
    N &= t - 2t^2
\end{align*}
\]

Now we can put all of these in the integral. Since \( t \) runs from 0 to 1, these are our limits.

\[
I = \int_C M \, dx + N \, dy = \int_0^1 t^4 \, dt + (t - 2t^2)2t \, dt = \int_0^1 t^4 + 2t^2 - 4t^3 \, dt = \frac{-2}{15}.
\]

(ii) Using the notation \( \int_C \mathbf{F} \cdot d\mathbf{r} \).

Again, we have to put everything in terms of \( t \):

\[
\begin{align*}
    \mathbf{F} &= (M, N) = (t^4, t - 2t^2) \\
    \frac{d\mathbf{r}}{dt} &= (1, 2t), \ \text{so} \ \ d\mathbf{r} = \frac{d\mathbf{r}}{dt} \, dt = (1, 2t) \, dt
\end{align*}
\]

Thus, \( \mathbf{F} \cdot d\mathbf{r} = (t^4, t - 2t^2) \cdot (1, 2t) \, dt = t^4 + (t - 2t^2)2t \, dt \). So the integral becomes

\[
I = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 t^4 + (t - 2t^2)2t \, dt.
\]

This is exactly the same integral as in method (i).

### 1.3 Work done by a force along a curve

Having seen that line integrals are not unpleasant to compute, we will now try to motivate our interest in doing so. We will see that the work done by a force moving a body along a path is naturally computed as a line integral.

Similar to integrals we’ve seen before, the work integral will be constructed by dividing the path into little pieces. The work on each piece will come from a basic formula and the total work will be the ‘sum’ over all the pieces, i.e. an integral.
1.3.1 Basic formula: work done by a constant force along a small line

We’ll start with the simplest situation: a constant force $\mathbf{F}$ pushes a body a distance $\Delta s$ along a straight line. Our goal is to compute the work done by the force.

The figure shows the force $\mathbf{F}$ which pushes the body a distance $\Delta s$ along a line in the direction of the unit vector $\mathbf{T}$. The angle between the force $\mathbf{F}$ and the direction $\mathbf{T}$ is $\theta$.

![Diagram showing force and displacement](image)

We know from physics that the work done by the force on the body is the component of the force in the direction of motion times the distance moved. That is,

$$\text{work} = |\mathbf{F}| \cos(\theta) \Delta s$$

We want to phrase this in terms of vectors. Since $|\mathbf{T}| = 1$ we know $\mathbf{F} \cdot \mathbf{T} = |\mathbf{F}| \cos(\theta)$. Using this in the formula for work we have

$$\text{work} = \mathbf{F} \cdot \mathbf{T} \Delta s. \quad (1)$$

Equation 1 is important and we will see it again. For now, we want to make one more substitution. We’ll call the vector $\Delta s \mathbf{T} = \Delta \mathbf{r}$. This is the displacement of the body. (Note, it is essentially the same as our formula $\frac{d\mathbf{r}}{dt} = \mathbf{T}$.) Using this, Equation 1 becomes

$$\text{work} = \mathbf{F} \cdot \Delta \mathbf{r}. \quad (2)$$

This is the basic work formula that we’ll use to compute work along an entire curve.

1.3.2 Work done by a variable force along an entire curve

Now suppose a variable force $\mathbf{F}$ moves a body along a curve $C$. Our goal is to compute the total work done by the force.

The figure shows the curve broken into 5 small pieces, the $j$th piece has displacement $\Delta \mathbf{r}_j$. If the pieces are small enough then the force on the $j$th piece is approximately constant. This is shown as $\mathbf{F}_j$. 

![Diagram showing variable force along curve](image)
If the pieces are small enough each segment is approximately a straight line and the force is approximately constant. So we can apply our basic formula for work and approximate the work done by the force moving the body along the \( j \)th piece as
\[
\Delta W_j \approx F_j \cdot \Delta r_j.
\]
The total work is the sum of the work over each piece.
\[
\text{total work} = \sum \Delta W_j \approx \sum F_j \cdot \Delta r_j.
\]
Now, as usual, we let the pieces get infinitesimally small, so the sum becomes an integral with is exactly the total work.
\[
\text{total work} = \int_C F \cdot dr.
\]
The subscript \( C \) indicates that it is the curve that has been split into pieces. That is, the total work is computed as a line integral of the force over the curve \( C \)!

1.3.3 Grad, curl and div

**Gradient.** For a function \( f(x, y) \): \( \text{grad} f = \nabla f = (f_x, f_y) \).

**Curl.** For a vector in the plane \( \mathbf{F}(x, y) = (M(x, y), N(x, y)) \) we define
\[
\text{curl} \mathbf{F} = N_x - M_y.
\]

**NOTE.** This is a scalar. In general, the curl of a vector field is another vector field. For vectors fields in the plane the curl is always in the \( \hat{k} \) direction, so we simply drop the \( \hat{k} \) and make curl a scalar. Sometimes it is called the ‘baby curl’.

**Divergence.** The divergence of the vector field \( \mathbf{F} = (M, N) \) is
\[
\text{div} \mathbf{F} = M_x + N_y.
\]

1.4 Properties of line integrals

In this section we will uncover some properties of line integrals by working some examples.

**Example GT.4.** First look back at the value found in Example GT.3. Now, use the same vector field as in that example, but, in this case, let \( C \) be the straight line from \((0, 0)\) to \((1, 1)\), i.e. same endpoints, but different path. Compute the line integral \( \int_C \mathbf{F} \cdot dr \).

**answer:** As always, start by sketching the curve:
We’ll use the notation $\int_C M \, dx + N \, dy$.

Parametrize the curve: $x = t$, $y = t$, with $t$ from 0 to 1.

Put everything in terms of $t$:

\[
\begin{align*}
dx &= dt \\
dy &= dt \\
M &= x^2y = t^3 \\
N &= x - 2y = -t
\end{align*}
\]

Now we put this into the integral

\[
I = \int_C M \, dx + N \, dy = \int_0^1 t^3 \, dt - t \, dt = \int_0^1 t^3 - t \, dt = -\frac{1}{4}.
\]

This is a different value from Example GT.3, which leads to the important principle:

**Important principle for line integrals.** Line integrals over two different paths with the same endpoints may be different.

**Example GT.5.** First look back at the value found in Example GT.3. Now, use the same vector field and curve as Example ?? except use the following (different) parametrization of $C$.

\[
x = \sin(t), \quad y = \sin^2(t); \quad 0 \leq t \leq \pi/2.
\]

Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

**answer:** We won’t sketch the curve it is identical to the one in Example GT.3. Putting everything in terms of $t$ we have

\[
\begin{align*}
dx &= \cos(t) \, dt \\
dy &= 2\sin(t) \cos(t) \, dt \\
M &= x^2y = \sin^2(t) \sin^2(t) = \sin^4(t) \\
N &= x - 2y = \sin(t) - 2\sin^2(t)
\end{align*}
\]

We put these in the integral $I = \int_C M \, dx + N \, dy$ and compute

\[
\begin{align*}
I &= \int_0^{\pi/2} \sin^4(t) \cos(t) \, dt + (\sin(t) - 2\sin^2(t))2\sin(t) \cos(t) \, dt \\
&= \int_0^{\pi/2} \left(\sin^4(t) + 2\sin^2(t) - 4\sin^3(t)\right) \cos(t) \, dt \\
&= \int_0^{\pi/2} \sin^4(t) \cos(t) \, dt + \int_0^{\pi/2} 2\sin^2(t) \cos(t) \, dt - 4\int_0^{\pi/2} \sin^3(t) \cos(t) \, dt \\
&= \int_0^1 u^4 + 2u^2 - 4u^3 \, du \\
&= \frac{2}{15}.
\end{align*}
\]
This is the same value we got in Example GT.3! In fact, the \( u \) substitution led to exactly the same integral! This leads us to the important principle:

**Important principle for line integrals.** The parametrization of the curve doesn’t affect the value of line the integral over the curve.

You should note that our work with work make this reasonable, since we developed the line integral abstractly, without any reference to a parametrization.

### 1.4.1 List of properties of line integrals

1. **Independent of parametrization:** The value of the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of the parametrization of \( C \).

2. **Reversing direction on the curve changes the sign:** If \( C \) is a curve, then we write \( -C \) for the same curve traversed in the opposite direction. In this case

   \[
   \int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}.
   \]

   (See the next example.)

**Example GT.6.** Let \( C \) be the curve from Example GT.3. Sketch \( C \) and \( -C \) and give a parametrization of \( -C \).

**answer:** \( C \) follows the parabola \( y = x^2 \) from \((0,0)\) to \((1,1)\), so the curve \(-C\) covers the same section of the parabola, but goes from \((1,1)\) to \((0,0)\).

The curve \( C \) can be parametrized as \( \mathbf{r}(t) = (t, t^2) \), with \( t \) running from \( 0 \) to \( 1 \). The easiest way to reverse this is to have \( t \) run from \( 1 \) to \( 0 \).

With this parametrization the \( t \) limits on the integral are reversed, which, we know from 18.01, changes the sign of the integral.

If you insist on an increasing parameter, we can parametrize \(-C\) by

\[
\mathbf{r}(u) = (1 - u, (1 - u)^2), \text{ with } u \text{ running from } 0 \text{ to } 1.
\]

3. **(Intrinsic formula)** We can write the line integral as

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds
\]

where \( \mathbf{T} = \text{unit tangent vector to } C \) and \( ds = \text{differential of arclength} \).
Reason: We know from our work on parametrized curves that \( \frac{dr}{dt} = T \frac{ds}{dt} \). So, \( dr = T \, ds \).

Alternatively, this is essentially Equations 1 and 2.

4. If \( C \) is a closed curve we use the notation

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C M \, dx + N \, dy. \]

The little circle on the integral sign indicates the curve is closed, i.e. starts and ends at the same point.

1.5 Rectangular and circular paths

Example GT.7. Evaluate \( I = \int_C y \, dx + (x + 2y) \, dy \) where \( C \) is the rectangular path from \((0,1)\) to \((1,0)\) shown below.

\[ \text{answer:} \] The path \( C \) is given in two pieces labeled \( C_1 \) and \( C_2 \). This means we will have to split the integral into two pieces, i.e.

\[ I = \int_C y \, dx + (x + 2y) \, dy = \int_{C_1} y \, dx + (x + 2y) \, dy + \int_{C_2} y \, dx + (x + 2y) \, dy. \]

We’ll do the integration one piece at a time. First, \( \int_{C_1} y \, dx + (x + 2y) \, dy \).

Parametrize \( C_1 \): We’ll use \( x \) as the parameter:

\[ x = x, \quad y = 1, \quad dx = dx, \quad dy = 0, \quad M = y = 1, \quad N = x + 2y = x + 2. \]

Put everything in terms of \( x \):

\[ x = x, \quad y = 1, \quad dx = dx, \quad dy = 0, \quad M = y = 1, \quad N = x + 2y = x + 2. \]

Put this in the integral and compute:

\[ \int_{C_1} M \, dx + N \, dy = \int_0^1 dx = 1. \]

Next, the integral over \( C_2 \).

Parametrize \( C_2 \): Use parameter \( y \): \( x = 1, \quad y = y, \quad y \) runs from \( 1 \) to \( 0 \).

Put everything in terms of \( y \):

\[ x = 1, \quad y = y, \quad dx = 0, \quad dy = dy, \quad M \, (\text{skip}), \quad N = x + 2y = 1 + 2y. \]
(We skipped $M$ because $dx = 0$.) Put this in the integral and compute

$$\int_{C_2} M \, dx + N \, dy = \int_1^0 1 + 2y \, dy = -2.$$

Adding, the pieces we have $I = 1 - 2 = -1$.

**Example GT.8.** Evaluate $I = \oint_C -y \, dx + x \, dy$ where $C$ is the unit circle traversed in a counterclockwise (CCW) direction.

**answer:** Parametrization: $x = \cos(t), \quad y = \sin(t), \quad 0 \leq t \leq 2\pi$. So, $dx = \cos(t) \, dt, \quad dy = -\sin(t) \, dt$. We get

$$I = \int_0^{2\pi} -\sin t (-\sin t) \, dt + \cos t (\cos t) \, dt = \int_0^{2\pi} \, dt = 2\pi.$$ 

1.6 Some super-duper, really seriously important examples

In these examples we are going to integrate a tangential field around a closed loop. These will be key computations as we explore Green’s Theorem and gradient fields.

In the following $r$ is the usual polar distance $r^2 = x^2 + y^2$.

**Example GT.9.** Let $\mathbf{F} = \left\langle \frac{-y}{r^2}, \frac{x}{r^2} \right\rangle$, and let $C$ be the unit circle traversed in a counterclockwise (CCW) direction. Compute $I = \oint_C \mathbf{F} \cdot d\mathbf{r}$

**answer:** Sketch $C$ and the vector field $\mathbf{F}$. 
Parametrize \( C: \ x = \cos(t), \ y = \sin(t), \ 0 \leq t \leq 2\pi. \)

Put everything in terms of \( t: \) (Note, on the unit circle \( r = 1. \))

\[
dx = \cos(t) \, dt, \quad dy = -\sin(t) \, dt, \quad M = -\frac{y}{r^2} = -\sin(t), \quad N = \frac{x}{r^2} = \cos(t).
\]

Put this in the integral and compute:

\[
I = \int_0^{2\pi} -\sin(t)(-\sin(t)) \, dt + \cos(t)(\cos(t)) \, dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t) \, dt = \int_0^{2\pi} dt = 2\pi.
\]

Example GT.10. Let \( \mathbf{F} \) be the same as the previous example. Let \( C_2 \) be the unit circle centered on \((2,0)\) traversed counterclockwise. Compute \( I_2 = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}. \)

**Answer:** Parametrize \( C_2: \ x = 2 + \cos(t), \ y = \sin(t), \ t \) from 0 to \( 2\pi. \)

Put everything in terms of \( t: \) (Note, \( r^2 \) is not constant.)

\[
dx = -\sin(t) \, dt \quad dy = \cos(t) \, dt \quad r^2 = x^2 + y^2 = (2 + \cos(t))^2 + \sin^2(t) = 5 + 4\cos(t) \quad M = -\frac{y}{r^2} = -\frac{\sin(t)}{5 + 4\cos(t)} \quad N = \frac{x}{r^2} = \frac{2 + \cos(t)}{5 + 4\cos(t)}
\]

Put this in the integral:

\[
I_2 = \int_{C_2} M \, dx + N \, dy = \int_0^{2\pi} \frac{\sin^2(t) + 2\cos(t) + \cos^2(t)}{5 + 4\cos(t)} \, dt = \int_0^{2\pi} \frac{1 + 2\cos(t)}{5 + 4\cos(t)} \, dt
\]

Oy! We put this into Wolfram Alpha and found \( I_2 = 0. \)

**Note.** We should suspect that the value of 0 is no accident. This is true and we will see this easily once we learn Green’s theorem. Avoiding actually computing an integral like this should be motivation enough for us to learn Green’s theorem.

**18.01 challenge.** Compute the integral for \( I_2. \) Hints: You can use the substitution \( u = \tan(t/2) \) and partial fractions. It’s best to use symmetry and compute 2 times the integral from 0 to \( \pi. \)

### 1.7 Gradient and conservative fields

We will now focus on the important case where \( \mathbf{F} \) is a gradient field. That is, for some function \( f(x,y), \)

\[
\mathbf{F} = \nabla f = (f_x, f_y).
\]

**Note.** We will learn to call \( f \) a **potential function** for \( \mathbf{F}. \)
1.7.1 The fundamental theorem for gradient fields

**Theorem GT.11.** (Fundamental theorem for gradient fields) Suppose that \( \mathbf{F} = \nabla f \) is a gradient field and \( C \) is any path from point \( P \) to point \( Q \). The fundamental theorem for vector fields says

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P) = f(x_1, y_1) - f(x_0, y_0).
\]

where \( P = (x_0, y_0) \) and \( Q = (x_1, y_1) \).

That is, for gradient fields the line integral depends only on the endpoints of the path and is independent of the path taken.

\[
\begin{align*}
\int_C F \cdot dr &= f(x, y) |_P^Q = f(Q) - f(P) = f(x_1, y_1) - f(x_0, y_0). \\
\end{align*}
\]

The proof of the fundamental theorem is given after the next example

**Example GT.12.** Let \( f(x, y) = xy^3 + x^2 \) and let \( C \) be the curve shown. Compute \( \mathbf{F} = \nabla f \) and compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \) two ways: (i) directly as a line integral, (ii) using the fundamental theorem.

\[
\begin{align*}
\text{answer: } F(x, y) &= \nabla f (f_x, f_y) = (y^3 + 2x, 3xy^2) \\
\text{(i) Parametrize } C: &\quad x = t, \quad y = 2t, \quad t \text{ runs from 0 to 1.} \\
\text{Write everything in terms of } t: &\quad dx = dt, \quad dy = 2dt, \quad M = y^3 + 2x = 8t^3 + 2t, \quad M = 3xy^2 = 12t^3.
\end{align*}
\]
Put all this into the integral and compute:

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (y^3 + 2x) \, dx + 3xy^2 \, dy = \int_0^1 (8t^3 + 2t) \, dt + 12t^3 \, 2t \, dt = \int_0^1 32t^3 + 2t \, dt = 9. \]

(ii) By the fundamental theorem

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(1,2) - f(0,0) = 9. \]

You decide which method is easier!

1.7.2 Proof of the fundamental theorem

**Proof.** The proof uses the definition of line integral together with the chain rule and the usual fundamental theorem of calculus.

We assume the following:
(i) \( \mathbf{F} = \nabla f \)
(ii) The curve \( C \) is parametrized by \( \mathbf{r}(t) = (x(t), y(t)) \), with \( t \) running from \( t_0 \) to \( t_1 \) and \( \mathbf{r}(t_0) = P, \mathbf{r}(t_1) = Q \).

First recall the for a parametrized curve \( \mathbf{r}(t) \) the chain rule says

\[ \frac{df(\mathbf{r}(t))}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt}. \]

So,

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = \int_{t_0}^{t_1} \nabla f \cdot \frac{d\mathbf{r}}{dt} \, dt = \int_{t_0}^{t_1} \frac{df(\mathbf{r}(t))}{dt} \, dt \]

\[ = f(\mathbf{r}(t_1))|_{t_0}^{t_1} = f(\mathbf{r}(t_1)) - f(\mathbf{r}(t_0)) = f(Q) - f(P) \quad \text{QED} \]

1.7.3 Path independence

**Definition.** For a vector field \( \mathbf{F} \) we say that the line integrals \( \int_C \mathbf{F} \cdot d\mathbf{r} \) are **path independent** if for any two points \( P \) and \( Q \) the integral yields the same value for *every* path connecting \( P \) to \( Q \).

From the fundamental theorem we can conclude: if \( \mathbf{F} = \nabla f \) is a gradient field, then the integrals \( \int_C \mathbf{F} \cdot d\mathbf{r} \) are path independent.

The following theorem offers an alternative way to express path independence.

**Theorem GT.13.** For a given vector field \( \mathbf{F} \), the line integrals \( \int_C \mathbf{F} \cdot d\mathbf{r} \) are path independent is equivalent to \( \oint_{C_c} \mathbf{F} \cdot d\mathbf{r} = 0 \) for any closed path \( C_c \).

**Proof.** To show equivalence we need to show two things:
(i) Path independence implies the line integral around any closed path is 0.
(ii) The line integral around all closed paths is 0 implies path independence.

Proof (i). To start, note that the constant path \( C_0 \) where \( \mathbf{r}(t) = P_0 \), with \( t \) running from 0 to 0 has line integral

\[
\oint_{C_0} \mathbf{F} \cdot d\mathbf{r} = \int_0^0 \mathbf{F} \cdot 0 \, dt = 0.
\]

Assume path independence and consider the closed path \( C_c \) shown in Figure (i) below. Since both \( C_c \) and \( C_0 \) have the same start and end points, path independence says the line integrals are the same, i.e.

\[
\oint_{C_c} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_0} \mathbf{F} \cdot d\mathbf{r} = 0.
\]

This proves (i).

(ii) Assume \( \oint_{C_c} \mathbf{F} \cdot d\mathbf{r} = 0 \) for any closed curve. Consider two paths between \( P \) and \( Q \) as shown in Figure (ii). The curve \( C_c = C_1 - C_2 \) is a closed curve starting and ending at \( P \).

Therefore, by assumption \( \oint_{C_c} \mathbf{F} \cdot d\mathbf{r} = 0 \). So

\[
0 = \int_{C_c} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}
\]

This implies \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \). That is, the integral is the same for all paths from \( P \) to \( Q \), i.e. the line integrals are path independent.

1.7.4 Conservative vector fields

Given a vector field \( \mathbf{F} \), Theorem GT.13 in the previous section said that the line integrals of \( \mathbf{F} \) were path independent is equivalent to the line integral of \( \mathbf{F} \) around any closed path is 0. Following physics terminology, we call such a vector field a conservative vector field.

The fundamental theorem says the if \( \mathbf{F} \) is a gradient field: \( \mathbf{F} = \nabla f \), then the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is path independent. That is,

\[
\text{a gradient field is conservative.}
\]
The following theorem says that the converse is also true on connected regions. By a connected region we mean that any two points in the region can be connected by a continuous path that lies entirely in the region.

**Theorem GT.14.** A conservative field on a connected region is a gradient field.

**Proof.** Call the region $D$. We have to show that if we have a conservative field $F = (M, N)$ on $D$ the there is a potential function $f$ with $F = \nabla f$.

The easy part will be defining $f$. The trickier part will be showing that its gradient is $F$. So, first we define $f$: Fix a point $(x_0, y_0)$ in $D$. Then for any point $(x, y)$ in $D$ we define

$$f(x, y) = \int_\gamma F \cdot dr,$$

where $\gamma$ is any path from $(x_0, y_0)$ to $(x, y)$.

Path independence guarantees that $f(x, y)$ is well defined, i.e. it doesn’t depend on the choice of path.

Now we need to show that $F = \nabla f$, i.e if $F = (M, N)$, then we need to show that $f_x = M$ and $f_y = N$. We’ll show the first case, the case $f_y = N$ is essentially the same. First, note that by definition

$$f_x(x, y) = \frac{df(x, y)}{dt} \bigg|_{t=0}.$$

This means we need to write down the line integral for $f(x + t, y)$.

In the figure below, $f(x, y)$ is the integral along the path $\gamma_1$ from $(x_0, y_0)$ to $(x, y)$ and $f(x + t, y)$ is the integral along $\gamma_1$ followed by the horizontal line $\gamma_2$ to $(x + t, y)$. This means that

$$f(x + t, y) = \int_{\gamma_1} F \cdot dr + \int_{\gamma_2} F \cdot dr = f(x, y) + \int_{\gamma_2} F \cdot dr.$$

We need to parameterize the horizontal segment $\gamma_2$. Notationwise, the letters $x$, $y$ and $t$ are already taken, so we let $\gamma_2(s) = (u(s), v(s))$, where

$$u(s) = x + s, \quad v(s) = y; \quad \text{with } 0 \leq s \leq t$$

On this segment $du = ds$ and $dy = 0$. So,

$$f(x + t, y) = f(x, y) + \int_0^t M(x + s, y) \, ds$$

The piece $f(x, y)$ is constant as $t$ varies, so the fundamental theorem of calculus says that

$$\frac{df(x + t, y)}{dt} = M(x + t, y).$$
Setting \( t = 0 \) we have
\[
fx(x, y) = \frac{df(x + t, y)}{dt} \bigg|_{t=0} = M(x, y).
\]
This is exactly what we needed to show! We summarize in a box:

"On a connected region a conservative field is a gradient field."

**Example GT.15.** If \( \mathbf{F} \) is the electric field of an electric charge it is conservative.

**Example GT.16.** A gravitational field of a mass is conservative.

### 1.8 Potential functions

**Definition.** If \( \mathbf{F} \) is a gradient field with \( \mathbf{F} = \nabla f \), then we call \( f \) a potential function for \( \mathbf{F} \).

**Note.** The usual physics terminology would be to call \(-f\) the potential function for \( \mathbf{F} \).

We have just seen that if \( \mathbf{F} = \nabla f \) (a gradient field) then \( \mathbf{F} \) is conservative. That is,

\[
\int_{P_0}^{P_1} \mathbf{F} \cdot d\mathbf{r}, \quad \text{is independent of the path from } P_0 \text{ to } P_1 \quad \text{and,} \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 0, \quad \text{where } C \text{ is any closed path.}
\]

**Questions:**
1. How can we tell if a vector field \( \mathbf{F} \) is a gradient field?
2. If \( \mathbf{F} \) is a gradient field, how do we find a potential function \( f \).

We start to answer these questions in the next theorem.

**Theorem GT.17.** Take \( \mathbf{F} = (M, N) \)

(a) If \( \mathbf{F} = \nabla f \) then \( \text{curl} \mathbf{F} = N_x - M_y = 0 \).

(b) If \( \mathbf{F} \) is differentiable and \( \text{curl} \mathbf{F} = 0 \) in the whole plane then \( \mathbf{F} = \nabla f \) for some \( f \), i.e. \( \mathbf{F} \) is conservative.

**Notes.** The restriction that \( \mathbf{F} \) is defined on the whole plane is too stringent for our needs. Below, we will give what we call the Potential Theorem, which only requires that \( \mathbf{F} \) be defined and differentiable on what’s called a simply connected region.

**Proof of (a):** If \( \mathbf{F} = (M, N) = \nabla f \) then we have \( M = f_x \) and \( N = f_y \). Thus,

\[
M_y = f_{xy}, \quad \text{and} \quad N_x = f_{yx}.
\]

This proves (a) (provided \( f \) has continuous second partials).

The proof of (b) will be postponed until after we have proved Green’s theorem and we can state the Potential Theorem. The examples below will show how to find \( f \).

**Example GT.18.** For which values of the constants \( a \) and \( b \) will \( \mathbf{F} = (axy, x^2 + by) \) be a gradient field?

**answer:** \( M_y = ax, \quad N_x = 2x \Rightarrow a = 2 \) and \( b \) is arbitrary.

**Example GT.19.** Is \( \mathbf{F} = ((3x^2 + y), e^x) \) conservative?

**answer:** \( M_y = 1, \quad N_x = e^x \). Since \( M_y \neq N_x \), \( \mathbf{F} \) is not conservative.
Example GT.20. Is \( \frac{(-y, x)}{x^2 + y^2} \) conservative?

**answer**: NO! The reasoning is a little trickier than in the previous example. First,

\[
M_y = \frac{y^2 - x^2}{(x^2 + y^2)^2} = N_x.
\]

BUT, since the field is not defined for all \((x, y)\) Theorem GT.17b does not apply.

The answer turns out to be no, \( F \) is not conservative. We can see that as follows. Consider \( C \) = unit circle parametrized by \( x = \cos t, y = \sin t \). Computing the line integral directly we have

\[
\oint_C F \cdot dr = \int_C -\frac{y}{x^2 + y^2} \, dx + \frac{x}{x^2 + y^2} \, dy = \int_0^{2\pi} dt = 2\pi.
\]

Since this is *not* 0 the field is not conservative.

Example GT.21. Is \( F = \frac{(x, y)}{x^2 + y^2} \) conservative?

**answer**: Again we find \( N_x = M_y \), BUT since \( F \) is not defined at \((0, 0)\) Theorem GT.17b does not apply. HOWEVER, it turns out that

\[
F = \nabla \ln(\sqrt{x^2 + y^2}) = \nabla \ln r
\]

Since we found a potential function we have shown directly that \( F \) is conservative on the region consisting of the plane minus the origin.

1.8.1 Finding the potential function

Example GT.22. Show \( F = (3x^2 + 6xy, 3x^2 + 6y) \) is conservative and find the potential function \( f \) such that \( F = \nabla f \).

**answer**: First, \( M_y = 6x = N_x \). Since \( F \) is defined for all \((x, y)\) Theorem GT.17 implies \( F \) is conservative.

**Method 1** (for finding \( f \)): Since \( \int_C F \cdot dr = f(P_1) - f(P_0) \) we have \( f(P_1) = f(P_0) + \int_C F \cdot dr \) for any path from \( P_0 \) to \( P_1 \).

We let \( P_1 = (x_1, y_1) \) be an arbitrary point. We can fix \( P_0 \) and \( C \) any way we want. For this problem take \( P_0 = (0, 0) \) and \( C = C_1 + C_2 \) as the path shown.
C_1 : x = 0, y = y, so \( dx = 0, dy = dy \),
C_2 : x = x, y = y_1, so \( dx = dx, dy = 0 \)

\[
f(x_1, y_1) - f(0,0) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M \, dx + N \, dy
\]

\[
= \int_0^{x_1} M(x,0) \, dx + \int_0^{y_1} N(x_1, y) \, dy
\]

\[
= \int_0^{y_1} 6y \, dy + \int_0^{x_1} 3x^2 + 6y_1 \, dx
\]

\[
= 3y_1^2 + x_1^3 + 3x_1^2 y_1
\]

Thus,

\[
f(x_1, y_1) - f(0,0) = 3y_1^2 + x_1^3 + 3x_1^2 y_1 = 3y_1^2 + x_1^3 + 3x_1^2 y_1.
\]

Switching from \((x_1, y_1)\) to \((x, y)\) and letting \(f(0,0) = C\), we have

\[
f(x, y) = 3y^2 + x^3 + 3x^2 y + C.
\]

**Method 2:** We start with \(f_x = 3x^2 + 6xy\).

Integrating with respect to \(x\) gives

\[
f(x, y) = x^3 + 3x^2 y + g(y).
\]

We need to add \(g(y)\) as the ‘constant’ of integration since \(y\) is a constant when differentiating with respect to \(x\).

Now integrating \(f_y = 3x^2 + 6y\) with respect to \(y\) gives

\[
f(x, y) = 3x^2 y + 3y^2 + h(x)
\]

Comparing the two expressions for \(f\) we see \(g(y) = 3y^2 + C\) and \(h(x) = x^3 + C\). So,

\[
f(x, y) = x^3 + 3x^2 y + 3y^2 + C.
\]

(The same as method 1.)

In general I prefer method 1.

**Example GT.23.** Let \(\mathbf{F} = ((x + y^2), (2xy + 3y^2))\). Show that \(\mathbf{F}\) is a gradient field and find the potential function using both methods.

**answer:** Testing the partials we have: \(M_y = 2y = N_x\), \(\mathbf{F}\) defined on all \((x, y)\). Thus, by Theorem GT.17, \(\mathbf{F}\) is conservative.

Method 1: Use the path shown.

\[
f(P_1) - f(0,0) = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M \, dx + N \, dy.
\]

C_1 : x = x, y = 0, \(\Rightarrow dx = dx, dy = 0 \Rightarrow M(x,0) = x\).

C_2 : x = x_1, y = y, \(\Rightarrow dx = 0, dy = dy \Rightarrow N(x_1, y) = 2x_1 y + 3y^2\).

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{x_1} x \, dx + \int_0^{y_1} 2x_1 y + 3y^2 \, dy = x_1^2 / 2 + x_1 y_1^2 + y_1^3.
\]
Thus, \( f(x_1, y_1) - f(0, 0) = \frac{x_1^2}{2} + x_1 y_1^2 + y_1^3 \). Equivalently

\[
\int f(x, y) = \frac{x^2}{2} + x y^2 + y^3 + C.
\]

Method 2: \( f_x = x + y^2 \), so \( f = \frac{x^2}{2} + x y^2 + g(y) \). \( f_y = 2xy + 3y^2 \), so \( f = xy^2 + y^3 + h(x) \).

Comparing the two expressions for \( f \) we get

\[
\int f(x, y) = \frac{x^2}{2} + x y^2 + y^3 + C.
\]

### 1.9 Green’s Theorem

Ingredients: \( C \) a simple closed curve (i.e. no self-intersection), and \( R \) the interior of \( C \). \( C \) must be positively oriented (traversed so interior region \( R \) is on the left) and piecewise smooth (a few corners are okay).

**Theorem GT.24.** Green’s Theorem: If the vector field \( \mathbf{F} = (M, N) \) is defined and differentiable on \( R \) then

\[
\oint_C M\, dx + N\, dy = \iint_R N_x - M_y\, dA. \tag{4}
\]

In vector form this is written

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}\mathbf{F} \, dA.
\]

where the curl is defined as \( \text{curl}\mathbf{F} = (N_x - M_y) \).

**Example GT.25.** Use the right hand side of Equation 4 to find the left hand side.

Use Green’s Theorem to compute

\[
I = \oint_C 3x^2y^2\, dx + 2x^2(1 + xy)\, dy
\]
where \( C \) is the circle shown.

**answer:** By Green’s Theorem

\[
I = \int_R \int 6x^2y + 4x - 6x^2y \, dA = 4 \int_R x \, dA.
\]

We could compute this directly, but here’s a trick. We know the \( x \) center of mass is

\[
x_{cm} = \frac{1}{A} \int_\mathcal{R} x \, dA = a.
\]

Thus \( \int_\mathcal{R} x \, dA = \pi a^3 \), so \( I = 4\pi a^3 \).

**Example GT.26.** (Use the left hand side of Equation 4 to find the right hand side.)

Use Green’s Theorem to find the area under one arch of the cycloid.

**answer:** The cycloid is \( x = a(\theta - \sin(\theta)) \), \( y = a(1 - \cos(\theta)) \). The picture shows the curve \( C = C_1 + C_2 \) surrounding the area under one arch.

By Green’s Theorem, \( \oint_C -y \, dx = \int_R \int dA = \text{area} \). So,

\[
\text{area} = \oint_{C_1+C_2} -y \, dx = \int_{C_1} 0 \cdot dx + \int_{C_2} -y \, dx
\]

\[
\text{area} = \int_0^{2\pi} a^2(1 - \cos(\theta))^2 \, d\theta = 3\pi a^2.
\]

**Other ways to compute area:**

\[
\int_\mathcal{R} dA = \oint_C -y \, dx = \oint_C x \, dy = \frac{1}{2} \oint_C -y \, dx + x \, dy.
\]
1.9.1 ‘Proof’ of Green’s Theorem

(i) First we’ll work on the rectangle shown. Later we’ll use a lot of rectangles to approximate an arbitrary region.

(ii) We’ll simplify a bit and assume \( N = 0 \). The proof when \( N \neq 0 \) is essentially the same with a bit more writing.

By direct calculation the right hand side of Green’s Theorem (Equation 4) is

\[
\iint_R - \frac{\partial M}{\partial y} \, dA = \int_a^b \int_c^d - \frac{\partial M}{\partial y} \, dy \, dx.
\]

Inner integral:

\[-M(x,y)|_c^d = -M(x,d) + M(x,c)\]

Putting this into the outer integral we have shown that

\[
\iint_R - \frac{\partial M}{\partial y} \, dA = \int_a^b M(x,c) - M(x,d) \, dx. \tag{5}
\]

For the left hand side of Equation 4 we have

\[
\oint_C M \, dx = \int_{bottom} M \, dx + \int_{top} M \, dx \quad \text{(since } dx = 0 \text{ along the sides)}
\]

\[
= \int_a^b M(x,c) \, dx + \int_b^a M(x,d) \, dx = \int_a^b M(x,c) - M(x,d) \, dx. \tag{6}
\]

Comparing Equations 5 and 6 we find that we have proved Green’s Theorem.

Next we’ll use rectangles to build up an arbitrary region. We start by stacking two rectangles on top of each other.

For line integrals when adding two rectangles with a common edge the common edges are traversed in opposite directions so the sum of the line integrals of the two rectangles equals the line integral over the outside boundary.
Similarly when adding a lot of rectangles: everything cancels except the outside boundary. This extends Green’s Theorem on a rectangle to Green’s Theorem on a sum of rectangles. Since any region can be approximated as close as we want by a sum of rectangles, Green’s Theorem must hold on arbitrary regions.

1.10 Extensions and applications of Green’s theorem

1.10.1 Simply connected regions

**Definition.** A region $D$ in the plane is **simply connected** if it has “no holes”. Said differently, it is simply connected for every simple closed curve $C$ in $D$, the interior of $C$ is fully contained in $D$.

**Examples:**

D1-D5 are simply connected. For any simple closed curve $C$ inside any of these regions the interior of $C$ is entirely inside the region.

**Note.** Sometimes we say the region is simply connected if any curve can be shrunk to a point without leaving the region.

The regions below are not simply connected. For each, the interior of the curve $C$ is not entirely in the region.
1.10.2 Potential Theorem

Here is a application of Green’s theorem which tells us how to spot a conservative field on a simply connected region. The theorem does not have a standard name, so we choose to call it the Potential Theorem. You should check that it is largely a restatement of Theorem GT.17 above.

**Theorem GT.27.** (Potential Theorem) Take \( \mathbf{F} = (M, N) \) defined and differentiable on a region \( D \).

(a) If \( \mathbf{F} = \nabla f \) then \( \text{curl} \mathbf{F} = N_x - M_y = 0 \).

(b) If \( D \) is simply connected and \( \text{curl} \mathbf{F} = 0 \) on \( D \), then \( \mathbf{F} = \nabla f \) for some \( f \).

We know that on a connected region, being a gradient field is equivalent to being conservative. So we can restate the Potential Theorem as: on a simply connected region, \( \mathbf{F} \) is conservative is equivalent to \( \text{curl} \mathbf{F} = 0 \).

**Proof of (a):** This was proved in Theorem GT.17.

**Proof of (b):** Suppose \( C \) is a simple closed curve in \( D \). Since \( D \) is simply connected the interior of \( C \) is also in \( D \). Therefore, using Green’s theorem we have,

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl} \mathbf{F} \, dA = 0.
\]

This shows that \( \mathbf{F} \) is conservative in \( D \). Therefore by Theorem GT.14 \( \mathbf{F} \) is a gradient field.

**Summary:** Suppose the vector field \( \mathbf{F} = (M, N) \) is defined on a simply connected region \( D \). Then, the following statements are equivalent.

1. \( \int_P^Q \mathbf{F} \cdot d\mathbf{r} \) is path independent.
2. \( \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \) for any closed path \( C \).
3. \( \mathbf{F} = \nabla f \) for some \( f \) in \( D \)
4. \( \mathbf{F} \) is conservative in \( D \).
If \( \mathbf{F} \) is continuously differentiable then 1, 2, 3, 4 all imply 5:

(5) \( \text{curl}\, \mathbf{F} = N_x - M_y = 0 \) in \( D \)

### 1.10.3 Why we need simply connected in the Potential Theorem

If there is a hole then \( \mathbf{F} \) might not be defined on the interior of \( C \). (See the example on the tangential field below.)

![Diagram of a simple closed curve with a hole](image)

### 1.10.4 Extended Green’s Theorem

We can extend Green’s theorem to a region \( \mathcal{R} \) which has multiple boundary curves.

Suppose \( \mathcal{R} \) is the region between the two simple closed curves \( C_1 \) and \( C_2 \).

![Diagram of a region with multiple boundary curves](image)

(Note \( \mathcal{R} \) is always to the left as you traverse either curve in the direction indicated.)

Then we can extend Green’s theorem to this setting by

\[
\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{R}} \text{curl}\, \mathbf{F} \, dA.
\]

Likewise for more than two curves:

\[
\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_3} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_4} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{R}} \text{curl}\, \mathbf{F} \, dA.
\]

**Proof.** The proof is based on the following figure. We ‘cut’ both \( C_1 \) and \( C_2 \) and connect them by two copies of \( C_3 \), one in each direction. (In the figure we have drawn the two copies of \( C_3 \) as separate curves, in reality they are the same curve traversed in opposite directions.)

Now the curve \( C = C_1 + C_3 + C_2 - C_3 \) is a simple closed curve and Green’s theorem holds on it. But the region inside \( C \) is exactly \( \mathcal{R} \) and the contributions of the two copies of \( C_3 \) cancel. That is, we have shown that

\[
\iint_{\mathcal{R}} \text{curl}\, \mathbf{F} \, dA = \iint_{C_1+C_3+C_2-C_3} \mathbf{F} \cdot d\mathbf{r} = \iint_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r}.
\]

This is exactly Green’s theorem, which we wanted to prove.
Example GT.28. Let \( \mathbf{F} = \left( -\frac{y}{r^2}, \frac{x}{r^2} \right) \) ("tangential field")
\( \mathbf{F} \) is defined on \( D = \text{plane} - (0,0) = \text{the punctured plane} \). (Shown below.)

It’s easy to compute (we’ve done it before) that \( \text{curl} \mathbf{F} = 0 \) in \( D \).

**Question:** For the tangential field \( \mathbf{F} \) what values can \( \oint C \mathbf{F} \cdot d\mathbf{r} \) take for \( C \) a simple closed curve (positively oriented)?

**answer:** We have two cases (i) \( C_1 \) not around 0 (ii) \( C_2 \) around 0

In case (i) Green’s theorem applies because the interior does not contain the problem point at the origin. Thus,
\[
\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \iint_{R} \text{curl} \mathbf{F} \, dA = 0.
\]

For case (ii) we will show that \( \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 2\pi \).

Let \( C_3 \) be a small circle of radius \( a \), entirely inside \( C_2 \). By the extended Green’s theorem we have
\[
\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} - \oint_{C_3} \mathbf{F} \cdot d\mathbf{r} = \iint_{R} \text{curl} \mathbf{F} \, dA = 0.
\]

Thus, \( \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_3} \mathbf{F} \cdot d\mathbf{r} \).

Using the usual parametrization of a circle we can easily compute that the line integral is
\[
\oint_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{2\pi} \, 1 \, dt = 2\pi. \quad QED.
\]
Answer to the question: The only possible values are 0 and 2π.

We can extend this answer in the following way:

If \( C \) is not simple, then the possible values of \( \int_C \mathbf{F} \cdot d\mathbf{r} \) are \( 2\pi n \), where \( n \) is the number of times \( C \) goes (counterclockwise) around \((0,0)\).

Not for class: \( n \) is called the winding number of \( C \) around 0. \( n \) also equals the number of times \( C \) crosses the positive \( x \)-axis, counting +1 from below and −1 from above.

Example GT.29. Let \( \mathbf{F} = r^n(x,y) \). Use extended Green’s theorem to show that \( \mathbf{F} \) is conservative for all integers \( n \). Find a potential function.

answer: First note, \( M = r^n x , \quad N = r^n y \). Thus, \( M_y = nr^{n-2} xy = N_x \). Said differently: \( \text{curl} \mathbf{F} = 0 \).

We can’t yet claim that \( \mathbf{F} \) is conservative because the origin is a possible problem point for \( \mathbf{F} \). So, we’ll show \( \mathbf{F} \) is conservative by showing \( \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \) for all simple closed curves \( C \).

Suppose \( C_1 \) is a simple closed curve not around 0 then Green’s theorem applies and we have \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0 \).

Next, suppose \( C_3 \) is a circle centered on \((0,0)\) then, since \( \mathbf{F} \) is radial and \( d\mathbf{r} \) is tangential to the curve we know \( \mathbf{F} \cdot d\mathbf{r} = 0 \). So, \( \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 0 \). (If you don’t like this argument, parametrize the circle in the usual way and show that the resulting integrand is 0.)

Now take the circle \( C_3 \) so large that it completely surrounds \( C_2 \). In this case, extended Green’s theorem implies (since \( \text{curl} \mathbf{F} = 0 \) between the curves) that

\[
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 0.
\]
Thus \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \) for all closed loops. As we know, this implies that \( \mathbf{F} \) is conservative.

To find the potential function we use method 1 over the curve \( C \) shown. The calculation works for \( n \neq -2 \). For \( n = -2 \) everything is the same except we get natural logs instead of powers. (We also ignore the fact that if \((x_1, y_1)\) is on the negative \(x\)-axis we should use a different path that doesn’t go through the origin. This isn’t really an issue since we already know a potential function exists, so continuity would handle these points without using an integral.)

Here’s the calculation of the potential:

\[
f(x_1, y_1) = \int_C r^n x \, dx + r^n y \, dy = \int_{-y_1}^{y_1} (1 + y^2)^{n/2} y \, dy + \int_{-x_1}^{x_1} (x^2 + y_1^2)^{n/2} x \, dx
\]

\[
= \left[ \frac{n + 2}{2} (1 + y_1^2)^{(n+2)/2} - 2(n+2)/2 \right]_{-y_1}^{y_1} + \left[ \frac{(x_1^2 + y_1^2)^{(n+2)/2}}{n + 2} - (1 + y_1^2)^{(n+2)/2} \right]_{-x_1}^{x_1}
\]

\[
= \frac{x_1^2 + y_1^2}{n + 2} - 2^{(n+2)/2} - (1 + y_1^2)^{(n+2)/2}
\]

Thus, our potential function is

\[
f(x, y) = \frac{r^{n+2}}{n+2} + C.
\]

In the special case of \( n = -2 \) we get \( f(x, y) = \ln r + C \).