Problem Set 3: Limits and closures

Your name:

Due: Thursday, February 18

Problem 1. Let $X$ be a topological space and $A, B \subseteq X$.

a. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

b. Show that $\overline{A \cap B} \subseteq \overline{A \cap B}$.

c. Give an example of $X, A$, and $B$ such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

d. Let $Y$ be a subset of $X$ such that $A \subseteq Y$. Denote by $\overline{A}$ the closure of $A$ in $X$, and equip $Y$ with the subspace topology. Describe the closure of $A$ in $Y$ in terms of $\overline{A}$ and $Y$.

Problem 2. Let $X$ be a set and let

$$\tau = \{U \in \mathcal{P}(X) : X \setminus U \text{ is finite, or } U = \emptyset\}.$$ 

a. Show that $\tau$ is a topology on $X$. This topology is called the cofinite topology (or finite complement topology).

b. Let $X$ be a set equipped with the cofinite topology. Let $A \subseteq X$. Describe the boundary $\partial A$ of $A$.

c. Suppose $X = \mathbb{N}$. To which points does the sequence $(n)_{n \in \mathbb{N}}$ converge?

Problem 3. Let $(X, d)$ be a metric space. Prove that the metric topology on $X$ is Hausdorff.

Problem 4. Let $X$ and $Y$ be topological spaces. A map $f : X \to Y$ is called open if for every open $U \subseteq X$, the image $f(U)$ is open in $Y$.

a. Consider $X \times Y$ equipped with the product topology. Show that the map $p_1 : X \times Y \to X, (x, y) \mapsto x$ is both continuous and open.

b. Consider $X \coprod Y$ equipped with the sum topology. Show that the map $i_1 : X \to X \coprod Y, x \mapsto (x, 0)$ is both continuous and open.

Problem 5. An equivalence relation on a set $X$ is a subset $R \subseteq X \times X$ such that

- for each $x \in X, (x, x) \in R$. 

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• for every $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.
• for every $x, y, z \in X$ if $(x, y), (y, z) \in R$ then $(x, z) \in R$.

We write $x \sim_R y$ as an abbreviation for $(x, y) \in R$ (and sometimes just write $x \sim y$). For $x \in X$, the set

$$[x] = \{y \in X : y \sim x\}$$

is called the equivalent class of $x$. We denote by

$$X/\sim = \{[x] : x \in X\},$$

the set of equivalence classes of elements of $X$, called the quotient of $X$ by $\sim$.

Suppose now that $X$ is a topological space with an equivalence relation $\sim$, and consider the map

$$\pi : X \to X/\sim, \ x \mapsto [x].$$

a. Declare a subset $U \subset X/\sim$ to be open if $\pi^{-1}(U) \subset X$ is open. Show that this defines a topology on $X/\sim$, and that the map $\pi$ is continuous. This topology is called the quotient topology.

b. Is the map $\pi$ always an open map? Justify your claim with proof or counterexample.

c. Let $Y$ be another topological space and let $f : X \to Y$ be a continuous map such that $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$. Show that there exists a unique map $\overline{f} : X/\sim \to Y$ such that $f = \overline{f} \circ \pi$, and show that $\overline{f}$ is continuous. This is called the universal property of the quotient topology.

d. Consider $\mathbb{R} \coprod \mathbb{R}$ with the sum topology, with the equivalence relation

$$(x, 0) \sim (y, 1) \text{ if and only if } x \neq 0 \text{ and } x = y.$$ 

The topological space $Q = \mathbb{R} \coprod \mathbb{R}/\sim$ is called the line with double origin. Which points in $Q$ are the limit of the sequence $n \mapsto [(\frac{1}{n+1}, 0)]$? Is $Q$ a Hausdorff space?