SINGULARITIES OF THE WAVE TRACE NEAR CLUSTER POINTS OF THE LENGTH SPECTRUM

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1. Introduction

Consider the Laplace operator Δ on the disk D,

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2; \quad D = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

and let u_j be the (real-valued, normalized) Dirichlet eigenfunctions

$$\Delta u_j = -\lambda_j u_j, x \in D; \quad u_j(x) = 0, \ x \in \partial D; \quad \int_D u_j(x)^2 dx = 1$$

with eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, indexed with multiplicity. The wave trace is the sum

$$h(t) = \sum_{j=1}^{\infty} e^{i\sqrt{\lambda_j}t},$$

which converges in the sense of distributions. The purpose of this article is to announce the following theorem.

Theorem 1.1. *$h(t)$, an infinitely differentiable function on $2\pi < t < 8$, has a finite limit and is infinitely differentiable at $t = 2\pi$ from the right.*

The significance of 2π is that it is a cluster point of the length spectrum from the left ($t < 2\pi$), as described in more detail below. It is easy to verify that there are no geodesics of length in between 2π and 8; it follows then from [GM] that $h(t)$ is smooth in $2\pi < t < 8$. The content of the theorem is that h is smooth from the right up to the endpoint 2π. The same proof applies to every cluster point $2\pi\ell$ of the length spectrum of geodesic flow on the disk.

We recall now the relationship between $h(t)$ and the wave equation. Consider the initial value problem for the wave equation,

$$\begin{aligned}(\partial_t^2 - \Delta)u(t,x) &= 0, \ x \in D, \ t > 0\\ u(t,x) &= 0, \ x \in \partial D, \ t > 0\\ u(0,x) &= f(x), \ x \in D\\ \partial_t u(0,x) &= g(x), \ x \in D\end{aligned}$$

The solution is

$$u(t,x) = \int_D K_t^{(1)}(x,y) f(y) dy + \int_D K_t^{(2)}(x,y) g(y) dy$$

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where

$$K_t^{(1)}(x,y)=\sum_{j=1}^{\infty}\cos(\sqrt{\lambda_j}t)u_j(x)u_j(y);\quad K_t^{(2)}(x,y)=\sum_{j=1}^{\infty}\frac{\sin(\sqrt{\lambda_j}t)}{\sqrt{\lambda_j}}u_j(x)u_j(y)$$

and the trace of the operator with kernel $K_t^{(1)}$ is

$$\int_D K_t^{(1)}(x,x)dx=\sum_{j=1}^{\infty}\cos(\sqrt{\lambda_j}t)=\operatorname{Re}h(t)$$

The same proof that shows that $h(t)$ is smooth as $t\to(2\pi)^+$ also shows that the trace

$$\int_D K_t^{(2)}(x,x)dx=\sum_{j=1}^{\infty}\sin(\sqrt{\lambda_j}t)/\sqrt{\lambda_j}$$

is smooth as $t\to(2\pi)^+$.

The close connection between between the length spectrum and the singularities of $h(t)$ was discovered by way of spectral and inverse spectral problems. In [CdV], Colin de Verdière showed that on a generic compact Riemannian manifold without boundary, the length spectrum is determined by the spectrum (list of eigenvalues of the Laplacian with multiplicity). Duistermaat and Guillemin [DG] and Chazarain [C] showed that the singular support of $h(t)$ is contained in the length spectrum and that the two sets are equal generically. Indeed, in the generic case, the singularity of $h(t)$ for t near L, the length of a geodesic, resembles a negative power of $t-L$, or, more precisely, a conormal distribution. In case the boundary is nonempty, Andersson and Melrose [AM] introduced the notion of generalized geodesic length spectrum and proved the analogous inclusion for the singular support of h. Subsequent work on inverse spectral problems for domains in the plane and, more generally, for manifolds with boundary can be found in [GM, CdV, Z, HeZ].

The periodic geodesics on the disk (with reflecting boundary) have lengths

$$L_{k,\ell}=k(2\sin(\pi\ell/k),\ \ell=\pm1,\pm2,\ldots,\ k=2,3,4,\ldots$$

with k the number of segments (or reflections) of the trajectory and ℓ the winding number of the trajectory around the origin. The degenerate cases, $(2,1)$, $(4,2)$, $(6,3)$, …, correspond to the trajectory that traverses a diameter 2, 4, 6, … times. It follows from [GM] that $h(t)$ is singular at $t=\pm L_{k,\ell}$ and smooth in the complement of the closure of these points. In particular, $h(t)$ is singular when t is the circumference of each regular polygon,

$$L_{2,1}<L_{3,1}<L_{4,1}<\cdots<L_{k,1}\to2\pi\quad k\to\infty$$

and h(t) is smooth in $2\pi<t<8$, since the shortest periodic geodesic with length greater than 2π has length $8=L_{4,2}$ (the 2-gon traced twice).

Thus the issue addressed here that is not addressed in previous works, is the behavior of $h(t)$ near a cluster point of the length spectrum. Although we examine only the case of the disk, which is far from generic, we expect the analogue of Theorem 1.1 to be valid for any convex domain in place of the disk. See the final remarks, below.

It is reasonable to conjecture that $h(t)$ has some power law (classical conormal) behavior as $t\to(2\pi)^+$. What is surprising is that the power is 0 and $h(t)$ is infinitely differentiable. The spikes at $t=L_{k,1}$ are asymmetrical and decay rapidly for $t>L_{k,1}$, so rapidly that an infinite sum of them with singularities at points closer and closer to 2π still converges at $t = 2\pi$. This is explained on a technical level by the fact that $h(t)$ is represented by sum of oscillatory integrals in which the first derivative of the phase function tends to infinity along one ray. Because the derivative of the phase is large, the phase changes quickly, and the corresponding integrals have more cancellation than one would obtain from a classical phase function.

2. Outline of the Proof

Our theorem is proved from systematic, optimal symbol properties of the zeros of Bessel functions (or, equivalently, the eigenvalues of the Laplace operator for the Dirichlet problem on the disk.

Let $\rho(m,n)$ denote the mth zero of the nth Bessel function J_n, $\rho(1,n) < \rho(2,n) < \cdots$. In polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, the eigenfunctions on the disk have the form

$$J_n(\rho(m,n)r)\cos n\theta, \quad J_n(\rho(m,n)r)\sin n\theta$$

with eigenvalue $-\lambda = -\rho(m,n)^2$. Since the multiplicity of the eigenvalue is two when $n \geq 1$ and one when $n = 0$,

$$h(t) = \sum_{(m,n)\in\mathbb{Z}^2} \psi_1(m)\psi_2(n)e^{it\rho(m,n)}$$

for any smooth cut-off functions ψ_1 and ψ_2 satisfying

$$\psi_1(m) = \begin{cases} 1, & m > 7/8 \\ 0, & m < 3/4 \end{cases} \quad ; \quad \psi_2(n) = \begin{cases} 0, & n < -1/4 \\ 1, & -1/8 < n < 1/8 \\ 2, & 3/4 < n \end{cases}$$

Below we will extend $\rho(m,n)$ in a natural way to be defined for real numbers m. (The extension to real numbers n will be the standard one for Bessel functions.) The Poisson summation formula then yields

$$h(t) = \sum_{(k,\ell)\in\mathbb{Z}^2} h_{k,\ell}(t)$$

where

$$h_{k,\ell}(t) = \int_{\mathbb{R}^2} \psi_1(m)\psi_2(n)e^{it\rho(m,n)-2\pi i(km+\ell n)}dm\,dn$$

Our main result, Theorem 1.1, follows immediately from

Theorem 2.1.

$$\left|\frac{d^{N_1}}{dt^{N_1}}h_{k,\ell}(t)\right| \leq C_{N_1,N_2}(1+|k|+|\ell|)^{-N_2}$$

for $2\pi < t < 2\pi + 1/10$

The domain of integration for $h_{k,\ell}$ is the quadrant $m \geq 3/4$, $n \geq -1/4$. We will deduce the estimates on $h_{k,\ell}$ from symbol estimates for $\rho(m,n)$.

In the sector range of the parameters, $m \geq cn$, for any fixed $c > 0$, the zeros of the Bessel functions satisfy ordinary symbol estimates as follows.

Proposition 2.2. *Fix a constant $c_0 > 0$. If $m \geq 3/4$, $n \geq -1/4$ and $m \geq c_0 n$, then*

a) $|\partial_m^j\partial_n^k\rho(m,n)| \leq C_{j,k}(m+n)^{1-j-k}$

b) $(\partial_m \rho, \partial_n \rho) = (\pi/\sin\alpha, \alpha/\sin\alpha) + O((m+n)^{-1})$ *as* $(m,n) \to \infty$ *where* α *is defined by*[1]

$$\tan\alpha - \alpha = \pi m/n$$

In the range of values of (m,n) complementary to Proposition 2.2, the appropriate symbol-type estimates involve fractional powers of m and n, and $\partial_m \rho$ does tend to infinity. (For m fixed $n \to \infty$ it turns out that $\partial_m \rho \approx n^{1/3}$. For our purposes the subtlest and most important bound will be the lower bound on the size of $\partial_n \rho$.)

Theorem 2.3. *There is an absolute constant* $c_1 > 0$ *such that if* $3/4 \le m \le c_0 n$ *(*c_0 *from Proposition 2.2) then*
a) $|\partial_m^j \partial_n^k (\rho(m,n) - n)| \le C_{j,k} m^{2/3-j}(m+n)^{1/3-k}$
b) $\partial_n(\rho(m,n) - n) \ge c_1 m^{2/3} n^{-2/3}$

Proposition 2.2 and Theorem 2.3 will be proved using asymptotic expansions of Bessel functions, which we derive by the method of Watson ([W], p. 251) starting from the Debye contour integral representation.

$$J_\nu(x) + iY_\nu(x) = \frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i} e^{x\sinh z - \nu z} dz \tag{2.1}$$

We cannot merely quote Watson's asymptotic expansion because we need to differentiate it. To some extent these differentiated estimates were carried out already by Ionescu and Jerison [IJ], but we need quite a bit more detailed asymptotics, especially in the transition region where $x - n \approx Cn^{1/3}$. Also, one needs to choose the right coordinate system since differentiation in some directions behaves differently from others.

Theorem 2.1 is proved by integration by parts. Consider the phase function of $h_{k,\ell}(t)$,

$$Q = i(t\rho(m,n) - 2\pi k m - 2\pi \ell n)$$

Since Q is smooth, the only issue is the asymptotic behavior as (m,n) tends to infinity. It is easy to show from Proposition 2.2 (b) that for large (m,n) the critical points of the phase Q occur near $t = L_{k,\ell}$ $(\alpha = \pi\ell/k)$.

The rest of the paper is organized as follows. In Section 3, we carry out the proof of Theorem 2.1, dividing the (m,n) quadrant of integration into two sectors. In the sector where $m \to \infty$, Proposition 2.2 and standard integration by parts and non-stationary phase methods apply. In the sector where $n \to \infty$, $3/4 \le m \le c_0 n$, the lower bound on $\partial_n Q$ given by Theorem 2.3 is used when integrating by parts with respect to n. On the other hand, when integrating by parts in the m variable, we will use the oscillation of e^{ikm} only and include the rest of the factors in the exponential e^Q in the amplitude.

In Section 4, we prove the symbol estimates and asymptotic expansions for Bessel function following the method of steepest descent (Debye contours). In Section 5, we deduce the symbol estimates for the zeros of Bessel functions stated above. We conclude with remarks about the relationship with earlier work and about the methods that can be expected to lead to the analogous result when the disk is replaced by a convex domain.

[1] We extend the definition of α continuously across $n = 0$, $\alpha = \pi/2$ by the reciprocal equation $1/(\tan\alpha - \alpha) = n/\pi m$.

3. Integration by Parts

We now deduce our main estimate, Theorem 2.1, from the symbol estimates for $\rho(m,n)$, Proposition 2.2 and Theorem 2.3. Denote

$$Q = i(t\rho(m,n) - 2\pi km - 2\pi\ell n)$$

Then

$$e^Q = \frac{-i}{t\partial_n\rho - 2\pi\ell}\partial_n e^Q \tag{3.1}$$

$$e^Q = e^{i(t\rho-2\pi\ell n}\frac{i}{2\pi k}\partial_m e^{-2\pi imk} \tag{3.2}$$

We will divide the region of integration into sectors. Consider first the "non-classical" region $3/4 \le m < c_0 n$. Define

$$I_{k,\ell}(t) = \int_{\mathbb{R}^2} \psi_1(m)\psi_1(cn/m)e^{i(t\rho(m,n)-2\pi km-2\pi\ell n)}dm\,dn$$

We focus at first on the case $\ell = 1$, $Q = i(t\rho(m,n) - 2\pi km - 2\pi n)$ from which all the singularities near $t = 2\pi$ arise. Applying formula 3.1 and integrating by parts,

$$\begin{aligned} I_{k,1}(t) &= \int \psi_1(m)\psi_1(cn/m)\frac{-i}{t\partial_n\rho - 2\pi}\partial_n e^Q dm\,dn \\ &= \int \frac{-it\partial_{nn}\rho}{(t\partial_n\rho - 2\pi)^2}\psi_1(m)\psi_2(cn/m)e^Q dm\,dn \\ &\quad + \int \frac{c}{m}\psi_1(m)\psi_1'(cn/m)\frac{i}{t\partial_n\rho - 2\pi}e^Q dm\,dn \end{aligned}$$

Repeating we obtain

$$\begin{aligned} I_{k,1}(t) &= c_1\int \frac{1}{m^2}\psi_1(m)\psi_1''(cn/m)\frac{1}{(t\partial_n\rho - 2\pi)^2}e^Q dm\,dn \\ &+ c_2\int \frac{1}{m}\psi_1(m)\psi_1'(cn/m)\frac{t\partial_{nn}\rho}{(t\partial_n\rho - 2\pi)^2}e^Q dm\,dn \\ &+ \int \psi_1(m)\psi_1(cn/m)\left[\frac{c_3(\partial_n^2\rho)^2}{(t\partial_n\rho - 2\pi)^4} + \frac{c_4 t\partial_n^3\rho}{(t\partial_n\rho - 2\pi)^3}\right]e^Q dm\,dn \end{aligned}$$

for appropriate coefficients $c_j(t)$ (polynomial in t). After N integrations by parts, $I_{k,1}$ is expressed as a linear combination of terms with integrand

$$e^Q\frac{(\partial_n^2\rho)^{a_2}(\partial_n^3\rho)^{a_3}\cdots(\partial_n^{N+1}\rho)^{a_{N+1}}}{(t\partial_n\rho - 2\pi)^{N+a_2+a_3+\cdots+a_{N+1}}}; \qquad (a_2 + 2a_3 + \cdots + Na_{N+1} = N)$$

times cutoff functions $\psi_1(m)\psi_1(cn/m)$ or derivatives of these cutoff functions. For $t \ge 2\pi$, the denominator has the lower bound

$$t\partial_n\rho - 2\pi \ge 2\pi(\partial_n\rho - 1) \ge cm^{2/3}n^{-2/3}$$

from Theorem 2.3(b). Moreover, Theorem 2.3(a) says, in particular, that

$$|\partial_n^j\rho| \lesssim m^{2/3}n^{1/3-j}$$

Denote $B = a_2 + a_3 + \cdots + a_{N+1}$. Then each of the integrands is bounded by

$$\begin{aligned}
\left|\frac{(\partial_n^2\rho)^{a_2}(\partial_n^3\rho)^{a_3}\cdots(\partial_n^{N+1}\rho)^{a_{N+1}}}{(t\partial_n\rho - 2\pi)^{N+a_2+a_3+\cdots+a_{N+1}}}\right| &\precsim \frac{(m^{\frac{2}{3}}n^{\frac{1}{3}-2})^{a_2}(m^{\frac{2}{3}}n^{\frac{1}{3}-3})^{a_3}\cdots(m^{\frac{2}{3}}n^{\frac{1}{3}-(N+1)})^{a_{N+1}}}{(m^{\frac{2}{3}}n^{-\frac{2}{3}})^{N+a_2+a_3+\cdots+a_{N+1}}} \\
&= \frac{m^{\frac{2}{3}B}n^{\frac{1}{3}B}n^{-(2a_2+3a_3+\cdots+(N+1)a_{N+1})}}{(m^{2/3}n^{-2/3})^N m^{\frac{2}{3}B}n^{-\frac{2}{3}B}} \\
&= \frac{n^{-(a_2+2a_3+\cdots+Na_{N+1})}}{(m^{2/3}n^{-2/3})^N} = \frac{n^{-N}}{(m^{2/3}n^{-2/3})^N} \\
&= m^{-\frac{2}{3}N}n^{-\frac{1}{3}N}
\end{aligned}$$

In particular, $I_{k,1}(t)$ is represented by a convergent integral.

Next, to prove rapid decay in k we integrate by parts in m using substitution 3.2. The first step is

$$\begin{aligned}
I_{k,1}(t) &= \int e^{i(t\rho-2\pi n)}\frac{-i}{2\pi k}\partial_m e^{-2\pi imk}\psi_1(m)\psi_1(cn/m)dm\,dn \\
&= \int \frac{-t\partial_m\rho}{2\pi k}e^Q\psi_1(m)\psi_1(cn/m)dm\,dn \\
&+ \int \frac{i}{2\pi k}e^Q\psi_1'(m)\psi_1(cn/m)dm\,dn + \int \frac{-cni}{2\pi m^2 k}e^Q\psi_1(m)\psi_1'(cn/m)dm\,dn
\end{aligned}$$

The term with the factor n/m^2 also has $\psi_1'(cn/m)$ so that it is supported where $cn \approx m$ and the term n/m^2 is comparable to $1/n \sim 1/m$. All terms have a gain of a factor $1/k$ except the one in which the derivative ∂_m falls on ρ. In that case,

$$|\partial_m\rho| \precsim n^{1/3}m^{-1/3}$$

In all, one step of type (3.2) yields the factor

$$\frac{n^{1/3}m^{-1/3}}{k}$$

Now we consider systematically what happens when steps of type (3.2) are applied after N steps of type (3.1). If the derivative δ_m in the integration by parts falls on a factor $\partial_n^j\rho$, then this gets replaced by $\partial_n^j\partial_m\rho/k$ and thus the bound is improved by the very favorable factor

$$\frac{1}{mk}$$

If the derivative ∂_m falls on the $e^{i(t\rho-2\pi n)}$ as in the first step, we have, as before a factor

$$\frac{n^{1/3}m^{-1/3}}{k}$$

If the derivative falls on a cutoff, then the gain is $1/k$. Finally, if the derivative falls on the denominator $(t\partial_n r - 2\pi)$ then it produces a factor

$$|\frac{\partial_n\partial_m\rho}{k(t\partial_n\rho - 2\pi)}| \precsim \frac{m^{-1/3}n^{-2/3}}{km^{2/3}n^{-2/3}} = \frac{1}{km}$$

In all, the worst case is the factor $n^{1/3}m^{-1/3}/k$ for each integration by parts in m. Thus if we integrate by parts N times in n and M times in m, the integrand will be bounded by

$$m^{-\frac{2}{3}N}n^{-\frac{1}{3}N}n^{\frac{1}{3}M}m^{-\frac{1}{3}M}k^{-M}$$

Therefore, if we choose N sufficiently large that

$$\frac{1}{3}N - \frac{1}{3}M > 2$$

then we obtain a convergent integrand that gives a bound on the integral by k^{-M}.

For $\ell \neq 1$, the bound is much simpler. $\partial_n \rho$ is very close to 1 for small m/n, so if $2\pi \leq t \leq 4\pi - \delta$ for any fixed $\delta > 0$, then the denominator in the integrands,

$$|t\partial_n \rho - 2\pi\ell| \approx |\ell|$$

for all integers $\ell \neq 1$. By integrating by parts N times, one finds the bound

$$|I_{k,\ell}| \precsim \ell^{-N}$$

for each N. The rapid decrease in k follows from similar reasoning to that given above for $I_{k,1}$.

Next, consider derivatives $(d/dt)^{N_1} I_{k,\ell}$. The integral representing this expression just has an extra factor of ρ^{N_1} in the integrand. This extra factor has size $(n^{1/3}m^{2/3})^{N_1}$ and symbol type bounds of the obvious kind after differentiation with respect to m and n. Thus one can compensate for these higher powers by more integrations by parts, and nearly the same proof as above shows that the derivatives of $I_{k,\ell}(t)$ are also rapidly decreasing in (k, ℓ). This ends the main portion of the proof.

What remains is to make estimates for the integrand in the region region $m > cn$ of integration.

$$h_{k,\ell} - 2I_{k,\ell} = \int [\psi_1(m)\psi_2(n) - 2\psi_1(m)\psi_1(cn/m)]e^Q dmdn \tag{3.3}$$

For this region we use Proposition 2.2. The informal idea is as follows. Fix $\alpha > 0$, and consider a ray in (m, n) space defined by

$$\tan\alpha - \alpha = \pi m/n$$

If $\nabla Q \to (0, 0)$ as $(m, n) \to \infty$ along this ray, then the asymptotic formula implies

$$t\pi/\sin\alpha = 2\pi k; \quad t\alpha \sin\alpha = 2\pi\ell,$$

Thus if Q has a "critical point near infinity" we can solve these equations for α and t and find

$$\alpha = \pi\ell/k; \quad t = 2k\sin(2\pi\ell/k) = L_{k,\ell}$$

This explains the singularities at $t = L_{k,\ell}$. The fact that the phase is nonstationary at other values of t will lead to a proof that $h_{k,\ell}(t)$ is smooth at each point $t \neq L_{k,\ell}$.

In more detail, first consider the range $|(m, n)| \leq C$, truncating the integrand with a smooth bump function in (m, n) variables. In that range, since the derivatives of ρ are bounded,

$$|\partial_n Q| \geq c|\ell|, \quad |\partial_m Q| \geq c|k|,$$

for sufficiently large $|k|$ and $|\ell|$. Hence, writing

$$e^Q = (1/\partial_n Q)\partial_n e^Q, \quad e^Q = (1/\partial_m Q)\partial_m e^Q,$$

and integrating by parts, one finds that the integral decays like $1/(|k| + |\ell|)$. Repeating N times, one finds that the integral decays like $O(1/(|k| + |\ell|)^N)$ for any N.

Next we turn to the range $|(m,n)| \geq C$, $m \geq cn$. In this case, we will also do integration by parts either in the variable m or n, and use lower bounds on $|\partial_n Q|$ or $|\partial_m Q|$.

Here we will use the asymptotic formula for $(\partial_m \rho, \partial_n \rho)$ of Proposition 2.2 b and restrict t to $2\pi \leq t \leq 2\pi + \delta$ for suitable small number δ.

First we confirm

$$|\partial_n Q| = |t\alpha/\sin\alpha - 2\pi\ell + O(1/(m+n))| \geq c(|\ell|+1), \tag{3.4}$$

Recall that α is defined by $\tan\alpha - \alpha = \pi m/n$. If $n > 0$, then, since $m \geq cn$, we have $\alpha_0 \leq \alpha \leq \pi/2$ for some fixed $\alpha_0 > 0$ depending on c. In the remaining range, $0 \geq n \geq -1/4$ and $m \geq C - 1/4$, which implies that $\pi/2 \leq \alpha \leq \pi/2 + \delta$ for some small δ of size on the order of $1/C$. It follows that

$$1 + \alpha_0^2/10 \leq \alpha/\sin\alpha \leq \pi/2 + 4\delta$$

The lower bound[2] on $\alpha/\sin\alpha$ bounds $|\partial_n Q|$ from below when $\ell \leq 1$ and the upper bound on $\alpha/\sin\alpha$ bounds $|\partial_n Q|$ from below in the case $\ell \geq 2$. Thus we have proved (3.4). We use this bound and integration by parts in n in the range. $|\ell| + C_2 \geq |k|$.

Lastly, if $k \geq |\ell| + C_2$, then (taking $C_2 = 10/\sin\alpha_0$) we have

$$|\partial_m Q| = |t\pi/\sin\alpha - 2\pi k + O(1/(m+n))| \geq c(|k|+1). \tag{3.5}$$

On the other hand, if $k \leq 0$, then (3.5) is obvious since $t\pi/\sin\alpha_0 \geq 2\pi^2$. In the range $|k| \geq |\ell| + C_2$, we use (3.5) and integration by parts in the m variable. This concludes the proof of Theorem 2.1.

4. Asymptotics of Bessel functions

In this section, we establish the optimal symbol properties of

$$H_n(x) := J_n(x) + iY_n(x)$$

as a functions of two variables (x, n). (The function $H_\nu(x)$, known as a Hankel function or Bessel function of the third kind, is denoted $H_\nu^{(1)}(x)$ in Watson's treatise [W] p. 73.) The asymptotic formula for the Bessel functions as the order and variable tend to infinity was discovered by Nicholson in 1910. In 1918, Watson used the Debye contour representation to give an appropriate bound on the error term. In his treatise on Bessel functions ([W] p. 249), Watson says of his own method that it is "theoretically simple (though actually it is very laborious)." To prove Theorem 2.3, we will carry out the even more laborious process of differentiating Watson's asymptotic formulas.

To state the symbol properties of $H_n(x)$ in the sector $n \leq x \leq 2n$, especially in the so-called transition region in which x is very close to n, will require a different coordinate system (β, ν). For $x \geq n > 0$, we write $\nu = n$ and define β by $x\cos\beta = n$. Define $a(\nu, \beta)$ by

$$H_\nu(\nu\sec\beta) = e^{i\nu(\tan\beta - \beta)} a(\nu, \beta) \tag{4.1}$$

Proposition 4.1. *If $\nu \geq 1/2$ and $0 \leq \beta \leq \pi/4$, and $a(\nu,\beta)$ is defined by (4.1), then*

$$|\partial_\nu^j \partial_\beta^k a| \precsim \nu^{-1/2-j} \beta^{-1/2-k}$$

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[2] The key here is that in the case $\ell = 1$, $t\alpha\sin\alpha - 2\pi \geq 2\pi\alpha_0^2/10 > 0$. We have avoided the critical points associated with $t = L_{k,1} \to 2\pi$. They occur at infinity along a rays in (m,n) space near the n axis, rays that are not in the sector $m \geq cn$.

We will also need more detailed asymptotics involving Airy functions. Define the function $A(y)$ for $y \in \mathbb{R}$ as the solution to the equation

$$A''(y) + 2yA(y) = 0$$

with initial conditions

$$A(0) = \Gamma(1/3)6^{-2/3}(3 + i\sqrt{3}); \quad A'(0) = \Gamma(2/3)6^{-1/3}(-3 + i\sqrt{3})$$

Proposition 4.2. *Denote* $b(\nu, \beta) = e^{-i\nu(\tan^3\beta)/3}\nu^{-1/3}A(y)$ *with* $y = (1/2)\nu^{2/3}\tan^2\beta$ *and* $a(\nu,\beta)$ *from Proposition 4.1. Then for* $0 \le \beta \le \pi/4$,

a) $|a - b| \precsim \nu^{-1}$.
b) $|\partial_\nu(a - b)| \precsim \nu^{-2}$.
c) $|\partial_\beta(a - b)| \precsim \nu^{-2/3}$ *[also* $\precsim 1/\nu\beta$ *if* $\beta \ge \nu^{-1/3}$*]*

Corollary 4.3. *If* $\nu >> 1$ *and* $0 \le \beta \le \pi/4$, *and* $a(\nu,\beta)$ *is defined by* (4.1), *then*

a) $|a| \approx \nu^{-1/2}\beta^{-1/2}$ *provided* $\nu^{-1/3} \le \beta \le \pi/4$.
b) $|a| \approx \nu^{-1/3}$ *provided* $\beta \le \nu^{-1/3}$.

In the remainder of the section, we will prove Propositions 4.1 and 4.2 and the corollary. The range, $x \ge 2n$ will be discussed at the end of the section.

Proof of Proposition 4.1. Define the phase function $\varphi(z)$ by

$$\varphi(z) = \frac{1}{\nu}(x \sinh z - \nu z) = \sec\beta \sinh z - z$$

The contour integral (2.1) can then be written

$$H_\nu(\nu \sec\beta) = \frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i} e^{\nu\varphi(z)}dz$$

The contour of steepest descent[3] passes through $z = i\beta$ and is parametrized by the two curves, $z = \zeta_1(r,\beta) + i\beta$ and $z = \zeta_2(r,\beta) + i\beta$, in which the functions $\zeta_j(r,\beta)$, $j = 1, 2$, solve

$$\varphi(\zeta + i\beta) - \varphi(i\beta) = (\sec\beta)\sinh(\zeta + i\beta) - \zeta - i\tan\beta = -r, \quad r > 0, \tag{4.2}$$

and satisfy $\zeta_1(0,\beta) = \zeta_2(0,\beta) = 0$, $\operatorname{Re}\zeta_1(r,\beta) \le 0$, $\operatorname{Re}\zeta_2(r,\beta) \ge 0$. Moreover, $\zeta_1(r,\beta) \to -\infty - i\beta$ and $\zeta_2(r,\beta) \to \infty + i(\pi - \beta)$ as $r \to \infty$. Thus

$$\begin{aligned} H_\nu(\nu\sec\beta) &= \frac{e^{\nu\varphi(i\beta)}}{\pi i}\int_{-\infty}^{\infty+\pi i} e^{\nu(\varphi(z)-\varphi(i\beta))}dz \\ &= \frac{e^{i\nu(\tan\beta-\beta)}}{\pi i}\int_0^\infty e^{-\nu r}(\partial_r\zeta_2(r,\beta) - \partial_r\zeta_1(r,\beta))dr, \end{aligned}$$

and we have derived the formula for $a(\nu,\beta)$,

$$a(\nu,\beta) = \frac{1}{\pi i}\int_0^\infty e^{-\nu r}(\partial_r\zeta_2(r,\beta) - \partial_r\zeta_1(r,\beta))dr \tag{4.3}$$

Lemma 4.4. *For* $j = 1, 2$,

$$|\partial_r\partial_\beta^k\zeta_j(r,\beta)| \precsim \begin{cases} r^{-1/2}\beta^{-1/2-k}, & r < \beta^3 \\ r^{-(2+k)/3}, & \beta^3 < r \end{cases}$$

[3] We follow [W] p. 244 and pp. 249–252, except that where Watson uses e^{-xr}, we use $e^{-\nu r}$ so our expressions differ from his by factors $x/\nu = \cos\beta$.

Lemma 4.4 follows in a straightforward way from implicit differentiation and induction, but the proof takes a few pages. Abbreviate by ζ the functions $\zeta_j(r,\beta)$ along the contour of steepest descent satisfying (4.2). Differentiating (4.2) with respect to β yields

$$[\cosh(i\beta+\zeta)-\cosh(i\beta)]\partial_\beta\zeta+(\tan\beta)\sinh(\zeta+i\beta)+i\cosh(\zeta+i\beta)-i\sec\beta=0$$

Therefore, using $\cosh(a+b)=\cosh a\cosh b+\sinh a\sinh b$ with $a=\zeta+i\beta$ and $b=-i\beta$,

$$\partial_\beta\zeta=\frac{-i(\cosh\zeta-1)\sec\beta}{\cosh(\zeta+i\beta)-\cosh(i\beta)}$$

Define

$$F_1(\zeta)=1/\sinh(\zeta/2);\quad F_2(\zeta,\beta)=1/\sinh(\zeta/2+i\beta);\quad G(\zeta,\beta)=(\cosh\zeta-1)(\sec\beta)/\zeta^2$$

Since

$$\cosh(i\beta+\zeta)-\cosh(i\beta)=2\sinh(\zeta/2)\sinh(i\beta+\zeta/2),$$

we may write

$$\partial_\beta\zeta=\frac{-i}{2}F_1F_2G\zeta^2 \tag{4.4}$$

Similarly,

$$\partial_r\zeta=\frac{-1}{2}F_1F_2\cos\beta \tag{4.5}$$

Lemma 4.5. *Along the contour of steepest descent $\zeta=\zeta_1(r,\beta)$ and $\zeta=\zeta_2(r,\beta)$, for all $k\geq 0$,*

a) $|\partial_\beta^k F_1(\zeta(r,\beta))|\precsim\begin{cases}|\zeta|^{-1}(\beta+|\zeta|)^{-k} & \text{for}\quad |\zeta|\leq 1\\ e^{-|\zeta|/2} & \text{for}\quad |\zeta|\geq 1\end{cases}$

b) $|\partial_\beta^k F_2(\zeta(r,\beta),\beta)|\precsim\begin{cases}(\beta+|\zeta|)^{-k-1} & \text{for}\quad |\zeta|\leq 1\\ e^{-|\zeta|/2} & \text{for}\quad |\zeta|\geq 1\end{cases}$

c) $|\partial_\beta^k G(\zeta(r,\beta),\beta)|\precsim\begin{cases}(\beta+|\zeta|)^{-k} & \text{for}\quad |\zeta|\leq 1\\ e^{|\zeta|}/|\zeta|^2 & \text{for}\quad |\zeta|\geq 1\end{cases}$

d) $|\partial_\beta^{k+1}\zeta(r,\beta)|\precsim\begin{cases}|\zeta|(\beta+|\zeta|)^{-k-1} & \text{for}\quad |\zeta|\leq 1\\ 1 & \text{for}\quad |\zeta|\geq 1\end{cases}$

To begin the proof of Lemma 4.5, denote by $D_\zeta F(\zeta,\beta)$ the derivative of $F(\zeta,\beta)$ with β fixed and let $D_\beta F(\zeta,\beta)$ represent the derivative of $F(\zeta,\beta)$ with ζ fixed. (This is to distinguish from the partial derivative $\partial_\beta F(\zeta(r,\beta),\beta)$ representing the derivative with r fixed.) We bound D_ζ and D_β derivatives of F_1, F_2 and G as follows. On the curve of steepest descent ($\zeta=\zeta_j(r,\beta)$, $j=1,2$)

$$|D_\zeta^p F_1(\zeta)|=|(d/d\zeta)^p F_1(\zeta)|\precsim\begin{cases}|\zeta|^{-k-1} & \text{for } |\zeta|\leq 1\\ e^{-|\zeta|/2} & \text{for } |\zeta|\geq 1\end{cases} \tag{4.6}$$

$$|D_\zeta^p D_\beta^s F_2(\zeta,\beta)|\precsim\begin{cases}(\beta+|\zeta|)^{-p-s-1} & \text{for } |\zeta|\leq 1\\ e^{-|\zeta|/2} & \text{for } |\zeta|\geq 1\end{cases} \tag{4.7}$$

$$|D_\zeta^p D_\beta^s G(\zeta,\beta)|\precsim\begin{cases}1 & \text{for } |\zeta|\leq 1\\ e^{|\zeta|}/|\zeta|^2 & \text{for } |\zeta|\geq 1\end{cases} \tag{4.8}$$

To prove (4.6) when $p = 0$, note that if $\zeta = \xi + i\eta$ and $\xi \geq 0$, then

$$|\sinh(\zeta/2)| \geq (e^{\xi/2} - 1)/2 \geq \begin{cases} \xi/4, & \xi \geq 0 \\ e^{\xi/2}/4, & \xi \geq 2 \end{cases}$$

Along the contour $\zeta = \xi + i\eta = \zeta_2(r, \beta)$, $0 \leq \eta \leq \pi - \beta$. (Recall that $0 \leq \beta \leq \pi/2$.) Furthermore, the slope of η as a function of ξ is at most $\sqrt{3}$ ([W] 8.32, p. 240) so that $\xi \geq |\zeta| - \pi$ and $\xi \geq |\zeta|/10$. It follows that $|F_1(\zeta)| = |1/\sinh(\zeta)| \leq 40|\zeta|^{-1}$ for all $\zeta = \zeta_2(r, \beta)$. For $|\zeta| \geq 20$, we have $\xi \geq 2$ and consequently $|F_1(z)| \leq 4e^{-\xi/2} \leq 4e^{\pi/2}e^{-|\zeta|/2}$. This proves (4.6) for $p = 0$ and $\zeta = \zeta_2(r, \beta)$. The other branch $\zeta = \zeta_1$ is similar, with the only difference that the curve ζ_1 is in a horizontal strip of the complex plane is below the ξ-axis: $-\beta \leq \eta = \operatorname{Im} \zeta_1 \leq 0$. The case of $p = 1, 2, \ldots$ are easy consequences of the same estimates. A very similar proof to the one for (4.6) gives (4.7) because

$$|i\beta + \zeta| \approx \beta + |\zeta|$$

along the contour. The estimate (4.8) is easy.

We can now carry out the proof of Lemma 4.5 by induction. Parts (a)–(c) for $k = 0$ are the same as (4.6–4.8) for $p = 0$. Part (d) for $k = 0$ ($k + 1 = 1$) follows from (a)–(c) for $k = 0$ and the formula (4.4).

For the induction step, assume (a)–(d) are valid up to k. Part (a), $\partial_\beta^{k+1} F_1(\zeta(r, \beta))$, is a linear combination of terms

$$(D_\zeta^p F_1)(\partial_\beta^{q_1} \zeta)(\partial_\beta^{q_2} \zeta) \cdots (\partial_\beta^{q_p} \zeta) \quad (q_1 + \cdots + q_p = k + 1;\ q_j \geq 1)$$

Part (b), $\partial_\beta^{k+1} F_2(\zeta(r, \beta), \beta)$, is a linear combination of terms

$$(D_\zeta^p D_\beta^s F_2(z, \beta))(\partial_\beta^{q_1} \zeta)(\partial_\beta^{q_2} \zeta) \cdots (\partial_\beta^{q_p} \zeta) \quad (q_1 + \cdots + q_p + s = k + 1;\ q_j \geq 1)$$

Part (c), $\partial_\beta^{k+1} G(\zeta(r, \beta), \beta)$, is a linear combination of terms

$$(D_\zeta^p D_\beta^s G(z, \beta))(\partial_\beta^{q_1} \zeta)(\partial_\beta^{q_2} \zeta) \cdots (\partial_\beta^{q_p} \zeta) \quad (q_1 + \cdots + q_p + s = k + 1;\ q_j \geq 1)$$

Since the induction hypothesis says that $\partial_\beta^q \zeta$ has the given bounds for all $1 \leq q \leq k + 1$, each of the factors has appropriate bounds and multiplying them yields the appropriate bounds for (a)–(c) with k replaced by $k + 1$.

Lastly, to prove the induction step for (d), observe that

$$\partial_\beta^{k+1}(\partial_\beta \zeta) = \partial_\beta^{k+1}[(-i/2) F_1 F_2 G \zeta^2]$$

is a linear combination of terms of the form

$$(\partial_\beta^{k_1} F_1(\zeta))(\partial_\beta^{k_2} F_2(\zeta, \beta))(\partial_\beta^{k_3} G)(\partial_\beta^{k_4} \zeta)(\partial_\beta^{k_5} \zeta)$$

with $k_1 + k_2 + \cdots + k_5 = k + 1$. Since $k_j \leq k + 1$, we have already proved the appropriate bounds on each factor. Multiplying them yields the correct bound for $\partial_\beta^{k+2} \zeta$. This ends the proof of Lemma 4.5.

To convert the implicit bounds of Lemma 4.5 to ones in terms of r and β, observe that for $\zeta = \zeta_j(r, \beta)$,

$$|\zeta| \approx \begin{cases} r^{1/2} \beta^{-1/2}, & r \leq \beta^3 \\ r^{1/3}, & \beta^3 \leq r \leq 1 \\ 1 + \log r, & 1 \leq r \end{cases}$$

In the range $r \geq 1$, there is also a more precise estimate along the contour, namely,

$$e^{|\zeta|} \approx e^{|\xi|} \approx r \qquad (\zeta = \xi + i\eta)$$

It follows that

$$|i\beta+\zeta| \approx \beta+|\zeta| \approx \begin{cases} \beta, & r \le \beta^3 \\ r^{1/3}, & \beta^3 \le r \le 1 \\ 1+\log r, & 1 \le r \end{cases}$$

With these upper and lower bounds on $|\zeta|$ and $\beta+|\zeta|$, one can rewrite Lemma 4.5 as

Lemma 4.6. *Along the contour of steepest descent $\zeta = \zeta_1(r,\beta)$ and $\zeta = \zeta_2(r,\beta)$, for all $k \ge 0$,*

a) $|\partial_\beta^k F_1(\zeta(r,\beta))| \precsim \begin{cases} r^{-1/2}\beta^{1/2-k} & r \le \beta^3 \\ r^{-\frac{1}{3}(k+1)} & \beta^3 \le r \le 1 \\ r^{-1/2} & 1 \le r \end{cases}$

b) $|\partial_\beta^k F_2(\zeta(r,\beta),\beta)| \precsim \begin{cases} \beta^{-(k+1)} & r \le \beta^3 \\ r^{-\frac{1}{3}(k+1)} & \beta^3 \le r \le 1 \\ r^{-1/2} & 1 \le r \end{cases}$

c) $|\partial_\beta^k G(\zeta(r,\beta),\beta)| \precsim \begin{cases} \beta^{-k} & r \le \beta^3 \\ r^{-k/3} & \beta^3 \le r \le 1 \\ r/(1+\log r)^2 & 1 \le r \end{cases}$

d) $|\partial_\beta^{k+1}\zeta(r,\beta)| \precsim \begin{cases} r^{1/2}\beta^{-3/2-k} & r \le \beta^3 \\ r^{-\frac{1}{3}k} & \beta^3 \le r \le 1 \\ 1 & 1 \le r \end{cases}$

It is now routine to confirm Lemma 4.4. Differentiate (4.5),

$$\partial_\beta^k \partial_r \zeta = (-1/2)\partial_\beta^k (F_1 F_2 \cos\beta)$$

The right-hand side is a linear combination of terms of the form

$$(\partial_\beta^{k_1} F_1)(\partial_\beta^{k_2} F_2)(\partial_\beta^{k_3}\cos\beta), \quad k_1+k_2+k_3 = k$$

which are bounded using Lemma 4.6. Now that Lemma 4.4 is proved, Proposition 4.1 follows from the formula (4.3) for $a(\nu,\beta)$.

We now turn to the proof of Proposition 4.2. The function φ from (4.2) can be rewritten

$$\varphi(z+i\beta) - \varphi(i\beta) = (i\tan\beta)(\cosh z - 1) + (\sinh z - z) \tag{4.9}$$

The Taylor approximations $\cosh z - 1 \approx z^2/2$ and $\sinh z - z = z^3/6$ so the cubic approximation to (4.9) is

$$\Phi(z) = (i\tan\beta)z^2/2 + z^3/6 \tag{4.10}$$

Following Watson, we prove Proposition 4.2 by comparing $\zeta_1(r,\beta)$ and $\zeta_2(r,\beta)$, the two solutions to (4.2), to the functions $Z_1(r,\beta)$ and $Z_2(r,\beta)$ solving the corresponding equation

$$\Phi(Z) = -r, \quad r \ge 0 \tag{4.11}$$

with $Z_1(0,\beta) = Z_2(0,\beta) = 0$, and $\operatorname{Re} Z_1(r,\beta) \le 0$, $\operatorname{Re} Z_2(r,\beta) \ge 0$.

Following Watson again, curve $Z_1(r) \to -\infty - i\tan\beta$ as $r \to \infty$, whereas $Z_2(r)$ is asymptotic to the ray whose argument is $\pi/3$ as $r \to \infty$. Thus we define

$$b(\beta,\nu) = \int_{-\infty - i\tan\beta}^{e^{i\pi/3}\infty} e^{\nu\Phi(Z)} dZ$$

In order to show that this function $b(\beta,\nu)$ is the same as the function b of Proposition 4.2, we evaluate it using the contour consisting of the two rays $Z = -i\tan\beta - \xi$ and $Z = -i\tan\beta + \xi e^{i\pi/3}$, $\xi \geq 0$, rather than the steepest descent contour defined using Z_1 and Z_2. With the parametrization given for these rays, and the change of variable $\xi = u/\nu^{1/3}$, one obtains

$$\begin{aligned} b(\beta,\nu) &= e^{-i(\nu\tan^3\beta)/3} \int_0^\infty e^{-\nu\xi^3/6}\left[\gamma e^{\gamma(\nu\xi\tan^2\beta)/2} + e^{-(\nu\xi\tan^2\beta)/2}\right] d\xi \\ &= e^{-i(\nu\tan^3\beta)/3}\nu^{-1/3} A((\nu^{2/3}\tan^2\beta)/2) \qquad (\gamma = e^{\pi i/3}) \end{aligned} \tag{4.12}$$

where $A(t)$ is defined by

$$A(t) = \int_0^\infty e^{-u^3/6}[\gamma e^{\gamma t u} + e^{-tu}] du$$

We claim that

$$A''(t) + 2tA(t) = 0$$

Indeed, for $\lambda \in \mathbb{C}$, define

$$F_\lambda(t) = \int_0^\infty e^{-u^3/6 + \lambda t u} du$$

Then

$$\begin{aligned} -1 &= \int_0^\infty (d/du) e^{-u^3/6+\lambda t u} du \\ &= \int_0^\infty (-u^2/2 + \lambda t) e^{-u^3/6+\lambda t u} du \\ &= (-1/2\lambda^2) F''_\lambda(t) + \lambda t F_\lambda(t) \end{aligned}$$

Hence,

$$-\frac{1}{2} F''_\lambda(t) + \lambda^3 t F_\lambda(t) = -\lambda^2$$

Then $A(t) = \gamma F_\gamma(t) + F_{-1}(t)$ and $\gamma^3 = -1$ give the equation $A''(t) + 2tA(t) = 0$, as desired.

Thus $A(t)$ is an Airy-type function, identified uniquely by its value and derivative

$$A(0) = \Gamma(1/3) 6^{-2/3} (3 + i\sqrt{3}); \quad A'(0) = \Gamma(2/3) 6^{-1/3}(-3 + i\sqrt{3})$$

Writing A in terms of its real and imaginary parts, $A(t) = u(t) + iv(t)$, we find that the Wronskian takes the constant value

$$u(t)v'(t) - u'(t)v(t) = u(0)v'(0) - u'(0)v(0) = \Gamma(1/3)\Gamma(2/3)\sqrt{3} = 2\pi \tag{4.13}$$

Having identified b with the function of Proposition 4.2, we can now proceed with the proof.

Lemma 4.7. *For $j = 1,\ 2$, $0 \leq \beta \leq \pi/4$,*

$$|\zeta_j| + |Z_j| \precsim \begin{cases} r^{1/2}\beta^{-1/2} & r < \beta^3 \\ r^{1/3} & r > \beta^3 \end{cases}$$

Lemma 4.7 is routine and the proof is omitted. In fact ζ_j grows more slowly than Z_j for for $r >> 1$ (like $\log r$), but we don't make use of this.

Lemma 4.8. *For $j = 1,\ 2$,*

a) $$|\varphi'(\zeta_j)| = |i\tan\beta\sinh\zeta_j + \cosh\zeta_j - 1| \gtrsim \begin{cases} r^{1/2}\beta^{1/2} & r < \beta^3 \\ r^{2/3} & \beta^3 \le r \le 1 \\ r & 1 \le r \end{cases}$$

b) $$|\Phi'(Z_j)| = |i\tan\beta Z_j + Z_j^2/2| \gtrsim \begin{cases} r^{1/2}\beta^{1/2} & r < \beta^3 \\ r^{2/3} & r > \beta^3 \end{cases}$$

Part (a) of this lemma is proved in [IJ] 9.15 and 9.16, p. 1072. The proof of part (b) is similar and is omitted.

Lemma 4.9. *For $j = 1,\ 2$, $0 \le \beta \le \pi/4$,*

$$|\zeta_j - Z_j| \le \begin{cases} r^{3/2}\beta^{-3/2} & r \le \beta^3 \\ r & \beta^3 \le r \le 1 \\ r^{1/3} & 1 \le r \end{cases}$$

Proof. Fix a small number $r_0 > 0$. For $0r \ge r_0$, the estimate follows immediately from Lemma 4.7. For this argument we will use subscripts on the constants, because some will depend on others. For $|z| \le 1$, $|\varphi(z) - \Phi(z)| \le C_1(\beta|z|^4 + |z|^5)$. Fix r, β and $j = 1$ or 2. We consider first the case $0 < r \le \beta^3$. Denote $w_j = \zeta_j(r,\beta) - Z_j(r,\beta)$ and $Z_j = Z_j(r,\beta)$. Our goal is to show that

$$|w_j| \le Cr^{3/2}\beta^{-3/2}$$

The root in the appropriate half-plane of

$$\varphi(w + Z_j) + r = 0$$

is $w = w_j$. We will show that for a suitable constant C to be chosen later, the curve

$$S = \{\varphi(w + Z_j) + r : |w| = Cr^{3/2}\beta^{-3/2}\}$$

encloses the origin. Thus the root w_j is inside. We will show that S enclosed the origin by showing that $\varphi(w + Z_j) + r - \Phi'(Z_j)w$ is suitably small. Recall that $|\Phi'(Z_j)| \ge c_1 r^{1/2}\beta^{1/2}$. It follows that the circle

$$S_0 = \{\Phi'(Z_j)w : |w| = Cr^{3/2}\beta^{-3/2}\}$$

has radius at least Cc_1r^2/β.

By Lemma 4.7, $|Z_j| \le C_2 r^{1/2}\beta^{-1/2}$. We will require that

$$Cr^{3/2}\beta^{-3/2} \le C_2 r^{1/2}\beta^{-1/2} \tag{4.14}$$

so that $|w + Z_j| \le 2C_2 r^{1/2}\beta^{-1/2}$ and

$$|\varphi(w + Z_j) - \Phi(w + Z_j)| \le C_1(\beta|w + Z_j|^4 + |w + Z_j|^5) \le C_3 r^2/\beta$$

with a constant C_3 depending only on C_1 and C_2 (using $r \le \beta^3$). Furthermore, since $F(Z_j) = -r$,

$$\Phi(w + Z_j) = -r + \Phi'(Z_j)w + (1/2)(i\tan\beta + Z_j)w^2 + w^3/6$$

and

$$|(i\tan\beta + Z_j)w^2/2 + w^3/6| \le C^2\beta^{-2}r^3 + C_2C^2r^{7/2}\beta^{-7/2} + C^3r^{9/2}\beta^{-9/2}$$

We will require

$$(4.15) \qquad C^2\beta^{-2}r^3 + C_2C^2r^{7/2}\beta^{-7/2} + C^3r^{9/2}\beta^{-9/2} \le (1/100)c_1Cr^2/\beta$$

Now our final requirement on C is the lower bound

$$(4.16) \qquad C_3 \le (1/4)c_1C$$

Combining these estimates, we have for $|w| = Cr^{3/2}\beta^{-3/2}$,

$$|\varphi(w+Z_j) + r - \Phi'(Z_j)w| \le C_3r^2/\beta + (1/100)c_1Cr^2/\beta \le (1/2)c_1Cr^2/\beta$$

so that

$$|\varphi(w+Z_j) + r| \ge (1/2)c_1Cr^2/\beta.$$

On the other hand $|\Phi'(Z_j)w| \ge c_1Cr^2/\beta$. So S is a loop surrounding the origin.

Finally, we check that all three requirements on C are satisfied. For (4.16) fix $C = 4C_3/c_1$. Now that C is fixed, we choose $r_0 > 0$ sufficiently small that the other two inequalities (4.14) and (4.15) are satisfied for all r, $0 < r \le r_0$, $0 < r \le \beta^3$ $(0 \le \beta \le \pi/4)$. The remaining case, $\beta^3 \le r \le r_0$ is similar and slightly simpler. It will be omitted. This concludes the proof of Lemma 4.9.

Lemma 4.10. *For $j = 1,\ 2$,*

a) $|\partial_r Z_j| \le \begin{cases} r^{-1/2}\beta^{-1/2} & r < \beta^3 \\ r^{-2/3} & r > \beta^3 \end{cases}$

b) $|\partial_r \zeta_j| \le \begin{cases} r^{-1/2}\beta^{-1/2} & r < \beta^3 \\ r^{-2/3} & \beta^3 \le r \le 1 \\ r^{-1} & 1 \le r \end{cases}$

c) $|\partial_r(\zeta_j - Z_j)| \le 1$ *all r*

Proof. Differentiating (4.11), we find

$$\Phi'(Z_j)\partial_r Z_j = -1; \quad \varphi'(\zeta_j)\partial_r\zeta_j = -1$$

and the bounds of Lemma 4.8 imply parts (a) and (b). For part (c),

$$\begin{aligned} \partial_r(Z_j - \zeta_j) &= \frac{\Phi'(Z_j) - \varphi'(\zeta_j)}{\Phi'(Z_j)\varphi'(\zeta_j)} \\ &= \frac{\Phi'(\zeta_j) - \varphi'(\zeta_j)}{\Phi'(Z_j)\varphi'(\zeta_j)} + \frac{\Phi'(Z_j) - \Phi'(\zeta_j)}{\Phi'(Z_j)\varphi'(\zeta_j)} \end{aligned}$$

Lemma 4.8 implies

$$|\Phi'(Z_j)\varphi'(\zeta_j)| \gtrsim \max(r\beta, r^{4/3})$$

The formula of Φ' and Lemmas 4.7 and 4.9 imply

$$|\Phi'(Z_j) - \Phi'(\zeta_j)| \precsim (\beta + |Z_j| + |\zeta_j|)|Z_j - \zeta_j| \precsim \begin{cases} r^{3/2}\beta^{-1/2} & r < \beta^3 \\ r^{4/3} & \beta^3 \le r \le 1 \\ r^{2/3} & 1 \le r \end{cases}$$

Hence, $|(\Phi'(Z_j) - \Phi'(\zeta_j))/\Phi'(Z_j)\varphi'(\zeta_j))| \precsim 1$ Lemma 4.7 implies

$$|\Phi'(\zeta_j) - \varphi'(\zeta_j)| \precsim \beta|\zeta_j|^3 + |\zeta_j|^4 \precsim \begin{cases} r^{3/2}\beta^{-1/2} & r < \beta^3 \\ r^{4/3} & \beta^3 \le r \end{cases}$$

Hence, similarly, $|(\Phi'(\zeta_j) - \varphi'(\zeta_j))/\Phi'(Z_j)\varphi'(\zeta_j)| \precsim 1$. This concludes Lemma 4.10.

Lemma 4.11. *For $j = 1,\ 2$,*

a) $|\partial_r \partial_\beta Z_j| \le \begin{cases} r^{-1/2}\beta^{-3/2} & r \le \beta^3 \\ r^{-1} & \beta^3 \le r \end{cases}$

b) $|\partial_r \partial_\beta \zeta_j| \le \begin{cases} r^{-1/2}\beta^{-3/2} & r < \beta^3 \\ r^{-1} & \beta^3 \le r \end{cases}$

c) $|\partial_r \partial_\beta (\zeta_j - Z_j)| \precsim r^{-1/3}$

Proof. Differentiate the implicit equation with respect to β to obtain

$$\partial_\beta Z_j = -(i \sec^2 \beta) Z_j^2 / 2\Phi'(Z_j)$$

and

$$\partial_\beta \partial_r Z_j = -(i \sec^2 \beta)\partial_r Z_j [2 Z_j \Phi'(Z_j) - F''(Z_j) Z_j^2]/2F'(Z_j)^2$$

We have already bounded each of these terms and the bounds combine to give Lemma 4.11 (a). To prove (b), differentiate (4.2) with respect to β to obtain

$$\partial_\beta \zeta_j = -i(\sec^2 \beta)(\cosh \zeta_j - 1)/\varphi'(\zeta_j)$$

Then, differentiating with respect to r,

$$\partial_r \partial_b \zeta_j = -i(\sec^2 \beta)\partial_r \zeta_j [\sinh \zeta_j \varphi'(\zeta_j) - \varphi''(\zeta_j)(\cosh \zeta_j - 1)]/\varphi'(\zeta_j)^2$$

The estimates above for ζ_j, $\partial_r \zeta_j$, $\varphi'(\zeta_j)$, and

$$\varphi''(\zeta_j) \precsim \begin{cases} \beta & r \le \beta^3 \\ r^{1/3} & \beta^3 \le r \le 1 \\ r & 1 \le r \end{cases}$$

combine to give part (b) of Lemma 4.11.

To prove (c) write

(4.17)
$$\partial_\beta(\zeta_j - Z_j) = -i \sec^2 \beta [Z_j^2(\varphi'(\zeta_j) - \varphi'(Z_j)) + Z_j^2(\varphi'(Z_j) - \Phi'(Z_j)) +$$

$$(Z_j - \zeta_j)^2 \Phi'(Z_j) - 2(\cosh \zeta_j - 1 - \zeta_j^2/2)\Phi'(Z_j)]/\Phi'(Zj)\varphi'(\zeta_j) \tag{4.18}$$

With z is a point on the line segment from Z_j to ζ_j,

(4.19)
$$|\partial_\beta(\zeta_j - Z_j)| \le C|Z_j|^2 |\varphi''(z)||Z_j - \zeta_j| + |Z_j|^2(\beta |Z_j|^3 +$$

$$|Z_j|^4) + |Z_j||Z_j - \zeta_j||\Phi'(Z_j)| + |\zeta_j|^4 ||\Phi'(Z_j)|]/|\Phi'(Z_j)\varphi'(\zeta_j)| \tag{4.20}$$

Using the preceding bounds and (4.19),

$$|\partial_\beta(\zeta_j - Z_j)| \precsim \begin{cases} r^{3/2}\beta^{-5/2} & r \le \beta^3 \\ r^{2/3} & \beta^3 \le r \le 1 \end{cases}$$

In the range $0 < r \le 1$, differentiation of (4.17) with respect to r replaces terms Z_j (with the bound $r^{1/2}\beta^{-1/2}$) by $\partial_r Z_j$ (with the bound $r^{-1/2}\beta^{-1/2}$) or the similar replacement of ζ_j with $\partial_r \zeta_j$. This results in a bound of the same type as (4.19) with an extra factor of $1/r$. In other words,

$$|\partial_r \partial_\beta(\zeta_j - Z_j)| \precsim \begin{cases} r^{1/2}\beta^{-5/2} & r \le \beta^3 \\ r^{-1/3} & \beta^3 \le r \le 1 \\ r^{-1} & 1 \le r \end{cases}$$

(The case $r \geq 1$ follows separately from parts (a) and (b).) In all three cases this is less than $r^{-1/3}$, so this concludes Lemma 4.11.

Define

$$b(\beta, \nu) = \int_0^\infty e^{-\nu r}(\partial_r Z_2(r, \beta) - \partial_r Z_1(r, \beta))dr$$

Then Lemmas 4.10 and 4.11 imply that

$$|a(\beta, \nu) - b(\beta, \nu)| \leq \int_0^\infty e^{-\nu r} dr \leq 1/\nu \tag{4.21}$$

$$|\partial_\nu [a(\beta, \nu) - b(\beta, \nu)]| \leq \int_0^\infty r e^{-\nu r} dr \leq 1/\nu^2 \tag{4.22}$$

$$|\partial_\beta [a(\beta, \nu) - b(\beta, \nu)]| \leq \int_0^\infty r^{-1/3} e^{-\nu r} dr \leq \nu^{-2/3} \tag{4.23}$$

This concludes the proof of Proposition 4.2.

Next, in order to deduce Corollary 4.3, we will prove some estimates for $b(\nu, \beta)$ that will also be needed in the next section.

Proposition 4.12. *If $y \geq 0$ and $\nu \geq 1/2$, $0 \leq \beta \leq \pi/4$, then*

a) $|b| \approx \begin{cases} \nu^{-1/3} & \beta \leq \nu^{-1/3} \\ \nu^{-1/2}\beta^{-1/2} & \beta \geq \nu^{-1/3} \end{cases}$

b) $|\partial_\nu b| \precsim \begin{cases} \nu^{-4/3} & \beta \leq \nu^{-1/3} \\ \nu^{-1/2}\beta^{5/2} & \beta \geq \nu^{-1/3} \end{cases}$

c) $|\partial_\beta b| \precsim \begin{cases} \nu^{1/3}\beta & \beta \leq \nu^{-1/3} \\ \nu^{1/2}\beta^{3/2} & \beta \geq \nu^{-1/3} \end{cases}$

Proof. The proposition will follow easily from the formula for b in terms of $A(y)$ and the estimates for $y \geq 0$,

$$|A(y)| \approx (1 + y)^{-1/4}; \qquad |A'(y)| \approx (1 + y)^{1/4} \tag{4.24}$$

(4.24) is well known, but we include a sketch of a proof. The upper bounds are standard; indeed, the asymptotic behavior as $y \to \infty$ follows from the fact that $A(y)$ is a multiple of the Hankel function $H_{1/3}((2y)^{3/2}/3)$ [W] p. 252. The lower bounds (for all $y \geq 0$) follow from the upper bounds and the fact that the Wronskian (4.13) is constant.

Next, using (4.24) we deduce (a). Recall that from (4.12),

$$|b(\nu, \beta)| = \nu^{-1/3}|A(y)| \approx \nu^{-1/3}(1 + |y|)^{-1/4}; \quad y = (1/2)\nu^{2/3}\tan^2\beta$$

When $\beta \leq \nu^{-1/3}$, $|y| \precsim 1$ and $|b| \approx \nu^{-1/3}$. When $\beta \geq \nu^{-1/3}$,

$$|b(\nu, \beta)| \approx \nu^{-1/3}(\nu^{2/3}\beta^2)^{-1/4} \approx \nu^{-1/2}\beta^{-1/2}$$

Parts (b) and (c) of Proposition 4.12 are proved as follows. Differentiating (4.12) gives

$$|\partial_\nu b(\nu, \beta)| \precsim \beta^3\nu^{-1/3}|A(t)| + \nu^{-2/3}\beta^2|A'(t)| \precsim \min(\nu^{-1/2}\beta^{5/2}, \nu^{-4/3})$$

and

$$|\partial_\beta b(\nu, \beta)| \precsim \nu^{2/3}\beta^2|A(t)| + \nu^{1/3}\beta|A'(t)| \precsim \min(\nu^{1/2}\beta^{3/2}, \nu^{1/3}\beta)$$

Proposition 4.12 (a) and (4.21) imply Corollary 4.3.

Finally, we discuss the range $\pi/4 \leq \beta < \pi/2$ and beyond. To formulate this we return to the variables (x, n). Fix $c > 0$ and let $\beta_0 > 0$ be the smallest number such that $\cos\beta_0 = 1/(1+c)$. Let β_1, $\pi/2 < \beta_1 < \pi$ satisfy $\cos\beta_1 = -1/4$. For $x \ge 1$, $n \ge -1/4$ and $x \ge (1+c)n$, define $\beta(x,n)$ as the unique number $\beta_0 \le \beta \le \beta_1$ such that $x\cos\beta = n$.

Proposition 4.13. *Let $x \ge 1$, $n \ge -1/4$ and $x \ge (1+c)n$ for a fixed $c > 0$. Define $\tilde{a}(x,n)$ by*

$$H_n(x) = e^{i(x\sin\beta - n\beta)}\tilde{a}(x,n).$$

with $\beta = \beta(x,n)$ defined above. Then

$$|\partial_x^j \partial_n^k \tilde{a}(x,n)| \precsim x^{-1/2-j-k}$$

To explain the connection with the previous notation, if $\nu = n$ and $x = \nu\sec\beta$, then $a(\nu,\beta) = \tilde{a}(x,n)$. The distinction between the coordinate systems is that ∂_ν is the derivative with β (or equivalently x/ν fixed), whereas ∂_n represents the derivative with x held fixed. The ∂_ν direction is special when β is near 0 and the ∂_n direction is special when β is near $\pi/2$. The estimates we carried out in the range u0 $\le \beta \le \pi/4$ can be extended to $\beta \to \pi/2$, but they require additional factors of $\sec\beta$ which tends to infinity. They do not suffice: In the eventual analysis of the behavior of zeros $\rho(m,n)$, the range $x = \rho(m,n) \ge (1+c)n$ corresponds to $m \ge cn$ and estimates in the (ν,β) coordinates give rise to error terms of size $O(1/n)$ when what is needed is $O(1/m) = O(1/\rho)$.

Proposition 4.13 was already proved in the case $n \ge 0$, $j = 0$, $k = 0,1,2$ in Theorem 9.1 (i) of [IJ]. (In the notation $a_x(\nu)$ of [IJ], $\nu = n$, $a_x(n)x^{-1/4}(x-n)^{-1/4} = \tilde{a}(x,n)$, and in the range of variables specified here, $x - n \approx x$.) The full proof of Proposition 4.13 follows the same procedure as in [IJ] pp. 1068–1072, with the only extra ingredient being the systematic treatment of derivatives of all orders, which was already carried out above in the very similar proof of the symbol estimates for $a(\nu,\beta)$ in Proposition 4.1 (a). These details will be omitted. We call attention to one difference. The integrals in the proof of Proposition 4.1 involve $e^{-\nu r}dr$ as $\nu \to \infty$, whereas in the proof of Proposition 4.13, (following [IJ]) the integrals involve $e^{-xr}dr$ and $x \to \infty$. We mention this in order to explain why the proof is unchanged when the range of n is extended from $n = \nu \ge 0$ to $n \ge -1/4$. The range $n \le 0$ would be problematic for integrals on $0 < r < \infty$ involving $e^{-nr}dr$, but these integrands are replaced by ones involving $e^{-xr}dr$ with $x \ge 1$.

5. Asymptotics of $\arg J_n + iY_n$ and of the zeros of Bessel functions

Denote $H_n(x) = J_n(x) + iY_n(x)$. It is well known that for all real numbers n, $|H_n(0^+)| = \infty$ and $|H_n(x)|$ is a decreasing function of x for $x > 0$ (Watson [W] p. 446). Moreover,

$$\partial_x \arg(H_n(x)) = \operatorname{Im}\frac{H_n'(x)}{H_n(x)} = \frac{J_n'(x)Y_n(x) - Y_n'(x)J_n(x)}{J_n(x)^2 + Y_n(x)^2} = \frac{2}{\pi x|H_n(x)|^2} > 0$$

Therefore, for all real values of n, as $x > 0$ increases, $H_n(x)$ traces a simple spiral counterclockwise in the complex plane. To choose a well-defined branch of the argument

$$\theta(x,n) = \arg(H_n(x))$$

note that $Y_n(x) = (J_n(x)\cos(n\pi) - J_{-n}(x))/\sin(n\pi)$ and one has the asymptotic formula $J_n(x) \sim (x/2)^n/\Gamma(n+1)$ as $x \to 0^+$. We deduce that for $n \ge 0$,

$$H_n(x)/|H_n(x)| \to -i = e^{-i\pi/2} \quad \text{as } x \to 0^+$$

whereas for $0 \leq n \leq -1/2$,

$$H_n(x)/|H_n(x)| \to -\sin n\pi - i\cos n\pi = e^{-i(n+1/2)\pi} \quad \text{as } x \to 0^+$$

Therefore, for $x > 0$ sufficiently small $H_n(x)$ is in the 4th quadrant. and a consistent definition of the branch is given by

$$\theta(0^+, n) = -\pi/2 \quad (n \geq 0)$$

and

$$\theta(0^+, n) = -(n + 1/2)\pi \quad 0 \geq n \geq -1/2$$

It then follows that the mth positive zero of $J_n(x)$ satisfies

$$\theta(\rho(m, n), n) = m\pi - \pi/2 \tag{5.1}$$

for $m = 1, 2, \ldots$. This implicit equation can then be used to extend the definition of $\rho(m, n)$ for all real m and n satisfying $m + n > 0$.

Next, we establish a few preliminary upper and lower bounds for $\rho(m, n)$. It follows from the fact that $\partial_x\theta(x, n) > 0$ that $\rho(m, n)$ well-defined and infinitely differentiable. In the range $|n| \leq 1/4$, the formula for $\partial_x\theta(x, n)$ and the estimates $|H_n(x)| \precsim x^{-1/4}$ in $0 < x < 1$ and $|H_n(x)| \precsim x^{-1/2}$ in $1 \leq x < \infty$ imply that

$$\rho(m, n) \approx m \quad (m \geq 1/2, \quad |n| \leq 1/4) \tag{5.2}$$

In the range $n \geq 1/4$, $m \geq 3/4$, we prove (well-known) upper and lower bounds

$$\rho(m, n) - n \approx m + m^{2/3}n^{1/3} \tag{5.3}$$

Let $n > 0$, then according to [W] (p. 485–487), $n < y_n$ where $y_n = \rho(1/2, n)$ is the smallest positive zero of $Y_n(x)$. Since $\theta(y_n, n) = 0$,

$$m\pi - \pi/2 = \int_{y_n}^{\rho(m,n)} \partial_x\theta(x, n)\, dx$$

For all $n \geq 1/4$, $|H_n(x)| \approx n^{-1/3}$ in $n \leq x \leq n + n^{1/3}$ and $|H_n(x)| \approx x^{-1/4}(x - n)^{-1/4}$ in $x \geq n + n^{1/3}$. These estimates along with the formula for $\partial_x\theta(x, n)$ above yield (5.3).

To prove Theorem 2.3, we require symbol estimates for $\theta(x, n)$ expressed in terms of the variables (β, ν) with $x = \nu\sec\beta$, $n = \nu$.

Lemma 5.1. *Denote*

$$\sigma(\nu, \beta) = \theta(\nu\sec\beta, \nu)$$

There is an absolute constant C such that if $C\nu^{-1/3} \leq \beta \leq \pi/4$, then

a) $|\partial_\nu^j\partial_\beta^k[\sigma(\beta, \nu) - \nu(\tan\beta - \beta)]| \precsim \nu^{-j}\beta^{-k} \quad j + k \geq 1$

b) $|\partial_\nu^j\partial_\beta^k[\sigma(\beta, \nu)| \precsim \nu^{1-j}\beta^{3-k} \quad j + k \geq 1$

In the transition region, where $c \leq \nu\beta^3 \leq C$, for some absolute constants $0 < c < C < \infty$ more detailed asymptotics are required. Denote $y = (\nu^{2/3}\tan^2\beta)/2$ and define the derivative of $\arg A(y)$ by

$$B(y) = \operatorname{Im}\frac{A'(y)}{A(y)} = \operatorname{Im}\frac{u(y)v'(y) - u'(y)v(y)}{|A(y)|^2}$$

where $A(y) = u(y) + iv(y)$ is the Airy function in (4.13). The asymptotic expansion for $\sigma(\nu, \beta)$ is given by

Lemma 5.2. *If $j + k \geq 1$, then*

a) $\partial_\nu \sigma = \dfrac{1}{3}(\nu^{-1/3} \tan^2 \beta) B(y)(1 + O(\beta^2 + \nu^{-4/3}\beta^{-2}))$

b) $\partial_\beta \sigma = (\nu^{2/3} \tan \beta \sec^2 \beta) B(y)(1 + O(\beta^2 + \nu^{-1}\beta^{-1}))$

Proof of Lemma 5.1. First note that the definition of σ and (4.1) imply

$$\partial_\beta \sigma = (\sec^2 \beta - 1)\nu + \operatorname{Im} \frac{\partial_\beta a}{a}$$

and

$$\partial_\nu \sigma = (\tan \beta - \beta) + \operatorname{Im} \frac{\partial_\nu a}{a}$$

Thus Lemma 5.1 follows from Proposition 4.1 and Corollary 4.3.

Next, we prove Lemma 5.2.

$$\begin{aligned}
\partial_\nu \sigma &= \tan \beta - \beta + \operatorname{Im}(\partial_\nu a/a) \\
&= \tan \beta - \beta + \operatorname{Im}(\partial_\nu b/b) + \operatorname{Im}[(\partial_\nu a - \partial_\nu b)/a] + \operatorname{Im}[(\partial_\nu b)(1/a - 1/b)] \\
&= \tan \beta - \beta + \operatorname{Im}(\partial_\nu b/b) + O(|\partial_\nu (a-b)|/|a| + |\partial_n b||a-b|/|ab|) \\
&= \tan \beta - \beta + \operatorname{Im}(\partial_\nu b/b) + O((\beta + \nu^{-1/3})^5)
\end{aligned}$$

Moreover (with $y = (\nu^{2/3} \tan^2 \beta)/2$), and recalling that $B(y) = \operatorname{Im}(A'(y)/A(y))$,

$$\operatorname{Im} \frac{\partial_\nu b}{b} = -\tan^3 \beta/3 + \frac{\nu^{-1/3}}{3} \tan^2 \beta B(y)$$

and $\tan \beta - \beta - (\tan^3 \beta)/3 \precsim \beta^5$. Thus,

$$\partial_\nu \sigma = \frac{\nu^{-1/3}}{3} \tan^2 \beta B(y) + O((\beta + \nu^{-1/3})^5)$$

Similarly,

$$\begin{aligned}
\partial_\beta \sigma &= \nu(\sec^2 \beta - 1) + \operatorname{Im}(\partial_\beta a/a) \\
&= \nu[\sec^2 \beta - 1 + \operatorname{Im}(\partial_\beta b/b) + \operatorname{Im}[(\partial_\beta a - \partial_\beta b)/a] + \operatorname{Im}[(\partial_\beta b)(1/a - 1/b)] \\
&= \nu[\sec^2 \beta - 1] + \operatorname{Im}(\partial_\beta b/b) + O(\nu(\beta + \nu^{-1/3})^4)
\end{aligned}$$

Moreover,

$$\operatorname{Im} \frac{\partial_\beta b}{b} = -\nu \tan^2 \beta \sec^2 \beta + \nu^{2/3} (\tan \beta)(\sec^2 \beta) B(y)$$

Since $\nu[\sec^2 \beta - 1] - \nu \tan^2 \beta \sec^2 \beta \precsim \nu \beta^4$,

$$\partial_\beta \sigma = \nu^{2/3} (\tan \beta)(\sec^2 \beta) B(y) + O(\nu(\beta + \nu^{-1/3})^4)$$

Finally, in order to write the error as a multiplicative expression, we use $|B(y)| \approx (1+y)^{1/2}$ which follows from the upper and lower bounds on $A(y)$ and $A'(y)$ (and ultimately from the Wronskian formula).

We can now deduce Theorem 2.3 from Lemmas 5.1 and 5.2. Consider the functions ν and β of n and m defined by

$$\nu = n; \quad \cos \beta = n/\rho(m,n)$$

For $3/4 \leq m \leq 3n$, $\rho(m,n) - n \approx m^{2/3} n^{1/3}$ implies

$$\beta(m,n) \approx (m/n)^{1/3} \quad \text{if } 3/4 \leq m \leq 3n$$

$$\nu \beta^2 \approx m^{2/3} n^{1/3} \implies \beta \approx (m/n)^{1/3}$$

In particular when $m \geq 3/4$, $\beta \geq cn^{-1/3}$ for some small absolute constant $c > 0$. We will be treating three ranges of β in different ways. One is the transition region where $cn^{-1/3} \leq \beta \leq Cn^{-1/3}$ where C is a large constant. The second is the region $Cn^{-1/3} \leq \beta \leq \beta_0$ for some small absolute constant β_0, and the third is the classical region $\beta_0 \leq \beta$. We will never need to consider β smaller than $cn^{-1/3}$.

Differentiating $\sigma(\nu, \beta) = m\pi - \pi/2$, we find the implicit formulas

$$\partial_n \beta = \frac{-\partial_\nu \sigma}{\partial_\beta \sigma}; \quad \partial_m \beta = \frac{\pi}{\partial_\beta \sigma}$$

We are going to prove by induction the property $Q(j, k)$ that says that for all $0 \leq \ell_1 \leq j$ and all $0 \leq \ell_2 \leq k$,

$$|\partial_m^{\ell_1} \partial_n^{\ell_2} \beta| \precsim \beta^{1-3\ell_1} \rho^{-\ell_1-\ell_2} (\approx m^{1/3-\ell_1} n^{-1/3-\ell_2})$$

$(\beta \approx (m/n)^{1/3})$ The property $Q(0, 0)$ is trivial. For $Q(0, 1)$, observe that by Lemma 5.2

$$\frac{\partial_\nu \sigma}{\partial_\beta \sigma} = \frac{\nu^{-1/3} \tan^2 \beta}{3\nu^{2/3} \tan \beta \sec \beta} (1 + O(\beta^2 + \nu^{-4/3} \beta^{-2})) \tag{5.4}$$

Moreover,

$$\partial_n \beta = O(\beta \nu^{-1}); \quad \partial_m \beta = O(\beta^{-2} \nu^{-1})$$

The proof of $Q(1, 0)$ is similar. Suppose that $Q(j, k)$ is valid. Differentiating the implicit formula for $\partial_m^j \partial_n^k \beta$ with respect to m, there are three types of things that can happen. First the derivative falls on the denominator, which is a power of $\partial_\beta \sigma$, in which case the expression is multiplied by a constant times

$$\frac{(\partial_\beta^2 \sigma) \partial_m \beta}{\partial_\beta s}$$

But recall that $|\partial_\beta \sigma| \succsim \beta^2 \nu$, $|\partial_\beta^2 \sigma| \precsim \beta \nu$, and $|\partial_m \beta| \precsim \beta^{-2} \nu^{-1}$ so that the product is majorized by $\beta^{-3} \nu^{-1}$. This is the new factor required for the estimate $Q(j+1, k)$. If the derivative falls on the numerator, then it may increase the degree of differentiation on a derivative of β, but always below the level of the induction hypothesis. Replacing a derivative of β by one derivative higher yields a change in estimation of the whole by the appropriate factor $\beta^{-3} \nu^{-1}$. Finally, the differentiation may land on a derivative of $D\sigma$. This replaces $D\sigma$ by $(\partial_\beta D\sigma) \partial_m \beta$, so the change in the estimation is the same (difference between $D\sigma$ and $\partial_\beta D\sigma$ is a factor β^{-1} and $\partial_m \beta$ is bounded by $\beta^{-2} \nu^{-1}$. Again the product is $\beta^{-3} \nu^{-1}$, which is the factor we want. Similarly, to prove $Q(j, k+1)$, differentiation with respect to n produces an estimate that differs by the factor ν^{-1} from the bound for $Q(j, k)$.

Next we can use $Q(j, k)$ to prove the property $P(j, k)$ that for all $0 \leq \ell_1 \leq j$ and all $0 \leq \ell_2 \leq k$,

$$|\partial_m^{\ell_1} \partial_n^{\ell_2} (\rho(m, n) - n)| \precsim \beta^{2-3\ell_1} \rho^{1-\ell_1-\ell_2} (\approx m^{2/3-\ell_1} n^{1/3-\ell_2})$$

In fact, implicit differentiation of $\rho \cos \beta = n$ with respect to n and with respect to m gives

$$\partial_n \rho - 1 = (\sec \beta - 1) + \rho (\tan \beta) \partial_n \beta$$

and

$$\partial_m \rho = \rho (\tan \beta) \partial_m \beta$$

The induction argument is similar to the proof of $Q(j, k)$ and is left to the reader. This concludes Theorem 2.3 (a).

We turn now to the lower bound (Theorem 2.3 b).

$$\begin{aligned}\partial_n\rho - 1 &= (\sec\beta - 1) + \rho(\tan\beta)\partial_n\beta \\ &= (\sec\beta - 1) - \rho(\tan\beta)\frac{\partial_\nu\sigma}{\partial_\beta\sigma} \\ &= \frac{1}{2}\beta^2 + O(\beta^4) - \frac{1}{3}\sin\beta\tan\beta(1 + O(\beta^2 + \nu^{-4/3}\beta^{-2}) \\ &= \frac{1}{6}\beta^2 + O(\beta^4 + \nu^{-4/3})\end{aligned}$$

Note that we need the precise asymptotics because the lower bound of order $\beta^2 \approx m^{2/3}n^{-2/3}$ requires the coefficient $1/2 - 1/3 = 1/6 > 0$. The error term is lower order if $\beta << 1$ and $n^{-4/3} << \beta^2$. Thus we have Theorem 2.3 (b) for n sufficiently large, in which case $\nu^{-2/3} << \beta \le C\nu^{-1/3}$. We do not need smaller values of β, because for $m \ge 1/20$, and β defined implicitly in terms of $\rho(m,n)$, $\beta \approx (m/n)^{1/3} >> \nu^{-2/3}$.

At last, here are some details of the much simpler estimates in what we are calling the classical region. Let $n \ge 1$ and $x \ge (1+c)n$. Then we are in the range in which $(x-n) \sim x$. As suggested in the remark of [IJ] p. 1069, following the methods above one finds

$$H_n(x) = x^{-1/4}(x-n)^{-1/4}e^{i(x\sin\beta - n\beta)}s(x,n)$$

where $\beta(x,n)$ is defined by $x\cos\beta = n$ and $s(x,n)$ satisfies

$$|\partial_x^j\partial_n^k s(x,n)| \precsim x^{-j-k}$$

Moreover, it is well-known that $|s(x,n)| > c > 0$. It follows that $\theta(x,n) = \arg H_n(x)$ (defined using any appropriate branch) satisfies

$$\theta(x,n) = x\sin\beta - n\beta + E(x,n)$$

where $E(x,n)$ is a symbol satisfying

$$|\partial_x^j\partial_n^k E(x,n)| \precsim x^{-j-k}$$

It is not hard to extend this estimate to negative values of n. Indeed the remaining range of n is the range $-1/2 \le n \le 1$. For a fixed range of the parameter n, the Hankel formula for $H_n(x)$ ([W] pp. 196–198) may be used. Let $x \ge 1$ and $-1/2 \le n \le x\cos\beta_0$, where β_0 is a small, fixed constant. Then since $\beta_0 > 0$, $x \ge (1+c)n$ for some $c > 0$ and we are in the range in which $(x-n) \simeq x$.

One calculates that

$$\partial_x\beta = \frac{n}{x^2\sin\beta}; \quad \partial_n\beta = -\frac{1}{x\sin\beta}$$

and

$$\partial_n\theta = -\beta + \partial_n E; \quad \partial_x\theta + \partial_x E$$

Differentiating the implicit equation for $\rho(m,n)$

$$\theta(\rho(m,n),n) = m\pi - \pi/2$$

with respect to m and n, we find

$$\partial_m\rho = \frac{\pi}{\partial_x\theta(\rho(m,n),n)} = \frac{\pi}{\sin\beta} + F_1(\rho,n)$$

and

$$\partial_n \rho = -\frac{\partial_n \theta(\rho(m,n),n)}{\partial_x \theta(\rho(m,n),n)} = \frac{\beta}{\sin\beta} + F_2(\rho,n)$$

with F_1 and F_2 satisfying

$$|\partial_\rho^j \partial_n^k F(\rho,n)| \precsim \rho^{-1-j-k}$$

and $\rho\cos\beta = n$, $\beta_0 \le \beta \le \pi/2 + \epsilon$ (β extends a small amount beyond $\pi/2$ to accomodate the negative values of n — we only care about ρ sufficiently large, so ϵ can be arbitrarily small.)

Finally, this implicit asymptotic formula for $\rho(m,n)$ can be differentiated many times expressing derivatives of $\rho(m,n)$ in terms of lower derivatives. This yields

$$|\partial_m^j \partial_n^k \rho(m,n)| \precsim \rho^{1-j-k} \approx m^{1-j-k}$$

for all $m \ge 100 + cn$ for a small fixed $c > 0$. For the asymptotic gradient we already have the implicit formula

$$(\partial_m \rho, \partial_n \rho) = (\pi/\sin\beta, \beta/\sin\beta) + F(\rho,n)$$

with F a symbol of order -1 as above. Thus of order $O(1/m)$ since $\rho \approx m$ in this range. We also have the implicit equation for ρ that gives us β as a function of ρ as follows:

$$n(\tan\beta - \beta) + E(\rho,n) = m\pi - \pi/2$$

so that ($\rho\cos\beta = n$)

$$\tan\beta - \beta = \frac{m\pi}{n}(1 - (E + \pi/2)/\pi m) = \frac{m\pi}{n}(1 - O(1/m))$$

Thus if we define $\alpha(m,n)$ by

$$\tan\alpha - \alpha = \pi m/n$$

(across $\pi/2$ as needed is perfectly ok) then $\alpha = \beta + O(1/m)$ and we get the asymptotic formula we wanted. (We can also characterize the error term as a symbol rather than with a bound, but we don't need this.)

6. Final Remarks

The length spectrum of the disk is far from generic. The regular k-gon trajectory on the disk can be rotated around the circle giving a one-parameter family of periodic geodesics of the same length, whereas in general the length of the geodesic with k reflections varies depending on where its vertices are. Nevertheless, we expect that the contribution to the wave trace at times t greater or equal to the perimeter from this family of k-reflection periodic trajectories resembles the effect of the family of regular k-gons in the case of the circle. Indeed the almost integrable behavior of the dyamical system of geodesic flow on convex planar regions near the generalized geodesic that follows the boundary has been treated in detail by Lazutkin [L] and by Melrose and Marvisi [MM]. Moreover, as we saw above, the asymptotics depended in a fundamental way on nondegeneracy conditions on derivatives of Airy functions. This is a hopeful sign, since general microlocal constructions of the parametrix of the wave equation near the boundary also involve Airy functions [AM, MS, MT].

As already mentioned, certain symbol properties of Bessel functions $J_\nu(x)$ as a function of the two variables x and ν were already proved in Ionescu and Jerison in [IJ]. There are three features of the treatment in [IJ] that are not enough for our purposes here. First, there is the minor point that we need bounds on derivatives of all orders, not just the first two. Second, the full symbol-type estimates (conjectured in [IJ], p. 1069) while valid are not sufficient here. Those bounds are for derivatives in the variables (x, ν), whereas for the present purpose in the range $\nu \leq x \leq 2\nu$, especially when x is near ν, it is necessary to distinguish a special directional derivative, namely, the derivative with respect to ν with the ratio x/ν held fixed (or equivalently with β defined by $\cos \beta = \nu/x$ held fixed). Third, [IJ] treats only upper bounds on $J_\nu(x) + iY_\nu(x)$ and its derivatives. We need both upper and lower bounds on the argument and its derivatives, which require detailed asymptotics, not just bounds on the Bessel functions and their first derivatives.

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