STRUCTURE OF ONE-PHASE FREE BOUNDARIES IN THE PLANE

DAVID JERISON AND NIKOLA KAMBUROV

Abstract. We study classical solutions to the one-phase free boundary problem in which the free boundary consists of smooth curves and the components of the positive phase are simply-connected. We show that if two components of the free boundary are close, then the solution locally resembles an entire solution discovered by Hauswirth, Hélein and Pacard, whose free boundary has the shape of a double hairpin. Our results are analogous to theorems of Colding and Minicozzi characterizing embedded minimal annuli, and a direct connection between our theorems and theirs can be made using a correspondence due to Traizet.

1. Introduction.

The one-phase free boundary problem in a disk $B \subseteq \mathbb{R}^2$,

$$\begin{array}{rll} u \geq 0 & \text{in} & B \\ \Delta u = 0 & \text{in} & B^+(u) := \{x \in B : u(x) > 0\} \\ |\nabla u| = 1 & \text{on} & F(u) := \partial B^+(u) \cap B \end{array} \tag{1}$$

arises as the Euler-Lagrange equation for the functional

$$I(u, B) = \int_B |\nabla u|^2 + 1_{\{u>0\}}\, dx \qquad u : B \to [0, \infty) \tag{2}$$

and appears in a variety of applications (e.g. jet flows in hydrodynamics, see [CS05]). The interior regularity theory of minimizers of the functional $I(u, B)$ with fixed boundary conditions on ∂B is well understood. Alt and Caffarelli [AC81] proved that the free boundary $F(u)$ is locally a graph of a C^∞ function (and hence analytic by [KN77]). Alt and Caffarelli also proved partial regularity of free boundaries in higher dimensions and established a strong analogy between the theory of free boundaries and the theory of minimal surfaces.

In keeping with [AC81] and many subsequent results ([ACF84, Caf87, Caf89, Wei98, CJK04, DSJ11, JS]) one should expect that most theorems about minimal surfaces have counterparts in the theory of free boundaries and vice versa. Here we consider classical solutions to (1) that are higher critical points rather than minimizers of the functional $I(u, B)$ with one additional purely topological assumption, namely that

$$\text{no connected component of } F(u) \text{ is compact in the open disk } B. \tag{3}$$

By classical solution we mean one for which $F(u)$ is a finite union of analytic curves. The topological assumption is equivalent to saying that the connected components of the positive phase are simply-connected. It is also equivalent to saying that the analytic curves, although they may become tangent at interior points, end at ∂B.

Our work is inspired by the groundbreaking work of Colding and Minicozzi on the structure of limits of sequences of embedded minimal surfaces of fixed genus in a ball in $\mathbb{R}^3$ ([CM04a, CM04b, CM04c, CM04d]). As it turns out, because of recent work of Traizet [Tra14], there is a direct overlap between our *a priori* estimates and rigidity results for families of solutions to (1) and the description of embedded minimal topological annuli due to Colding and Minicozzi.

Our starting place is the family of simply-connected planar regions $\Omega_a = a\Omega_1$, discovered by Hauswirth, Helein, and Pacard [HHP11], which solve the free boundary problem (1). They are defined by

$$\Omega_a := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1/a| < \pi/2 + \cosh(x_2/a)\}, \quad a > 0.$$

2010 *Mathematics Subject Classification.* 35R35, 35N25, 35Q35, 35BXX, 49Q05.

Key words and phrases. one-phase free boundary problem, overdetermined elliptic problem, minimal surfaces, compactness and singular limits.

The first author was supported by NSF grant DMS 1069225 and the Stefan Bergman Trust.

The boundary $\partial\Omega_a$ consists of two curves that we will refer to as hairpins. Hauswirth et al found a positive harmonic function $H_a(x) = aH_1(x/a)$ on Ω_a that satisfies the free boundary conditions $H_a = 0$ and $|\nabla H_a| = 1$ on $\partial\Omega_a$. Extending H_a to be zero in the complement of Ω_a, we have an entire solution to (1). (See Section 10 for the explicit formula for H_a using conformal mapping.)

Our first main result characterizes blow-up limits of classical solutions with simply-connected positive phase.

Theorem 1.1. *Let u_k be a sequence of classical solutions of* (1) *in the disk $B_{R_k} = B_{R_k}(0)$, with radius $R_k \nearrow \infty$, satisfying $0 \in F(u_k)$ and* (3). *Then a subsequence converges uniformly on compact subsets of $\mathbb{R}^2$ to some rigid motion of one of the following*

(a) $P(x) := x_2^+$, *a* half-plane solution*,*
(b) $W_b(x) := x_2^+ + (x_2 + b)^-$, *for some $b \geq 0$, a* two-plane solution*, or*
(c) $H_a(x)$*, for some $a > 0$, a* hairpin solution *as mentioned above and defined in* (27) *of Section 10.*

Note that unlike property (3), connectivity of the positive phase is not inherited in the limit. For example, blow-up limits of suitable translates and dilates of H_1 are two-plane solutions.

Theorem 1.1 is closely related to earlier classifications of entire solutions with simply-connected positive phase due to Khavinson, Lundberg and Teodorescu [KLT13] and Traizet [Tra14]. Traizet showed that classical entire solutions satisfying (3) must be of the form (a), (b), or (c). Khavinson et al showed that the same conclusion is true under a natural, weak regularity assumption on the free boundary known as the Smirnov property. We were not able to use this result to prove our theorem, and this is a central technical difficulty of the paper. Instead, we define another notion of weak solution that we can show is preserved under blow-up limits. Our weak solutions will satisfy both the properties of non-degenerate viscosity solutions introduced by L. Caffarelli and variational solutions introduced by G. Weiss. This PDE-theoretic approach has the benefit that it does not rely on complex function theory and so it could conceivably be extended to a higher-dimensionsional setting.

Our next result says that near points where the curvature of the free boundary is large, the boundary resembles a double hairpin.

Theorem 1.2. *Given $\delta > 0$ there exist positive numbers r, κ, ϵ and ϵ_1 with $0 < \epsilon_1 < \epsilon/2 < 1/100$, and an integer $N_0 \geq 0$ such that if u is a classical solution of* (1) *in B_1, satisfying* (3)*, then there are $N \leq N_0$ points $\{z_j\}_{j=1}^N \subseteq B_{3/4}$, with the properties:*

(a) The curvature of $F(u)$ is less than κ at any point of $F(u) \cap \Big(B_{1/2} \setminus \bigcup_{j=1}^N B_r(z_j)\Big)$.
(b) Near z_j, u is approximated by a hairpin solution, i.e. there exists some $a_j < \epsilon_1 r$ such that

$$|u(z_j + x) - H_{a_j}(\rho_j x)| \leq \delta a_j \quad \textit{for all} \quad |x| \leq 2a_j/\epsilon$$

for some rotation ρ_j.
(c) In $B_{2r}(z_j)\backslash B_{a_j/\epsilon}(z_j)$ the free boundary consists of four curves which are graphs in some common direction with small Lipschitz norms. More precisely, there exist f, $g : \mathbb{R} \to \mathbb{R}$ such that $f < g$,

$$\|f\|_{L^\infty} + \|g\|_{L^\infty} \leq \delta r, \quad \|f'\|_{L^\infty} + \|g'\|_{L^\infty} \leq \delta,$$

and

$$\{u = 0\} \cap (B_{2r}(z_j)\backslash B_{a_j/\epsilon}(z_j) = z_j + \rho_j(\{x : f(x_1) \leq x_2 \leq g(x_1)\} \cap \{x : a_j/\epsilon < |x| < 2r\})$$

The proof of parts (a) and (b) of Theorem 1.2 follow from the classification of blow-up solutions in Theorem 1.1. The proof of part (c) uses conformal mapping and is of independent interest. The usual flat $\implies$ Lipschitz step in regularity theory implies that the boundaries are Lipschitz graphs with small Lipschitz constant separately on each dyadic annulus, $2^{k-1} < |x - z_j| < 2^k$ for $a_j/\epsilon_0 < 2^k < r_0$. What part (c) rules out is the possibility of a spiral. It can be viewed as a quantitative version of the flat $\implies$ Lipschitz step, in which no information is used about the solution in a neighborhood $|x - z_j| < 50a_j$. Colding and Minicozzi call the analogous bound in the setting of minimal surfaces an effective *removable singularities theorem* [CM04c, Theorem 0.3]. This crucial estimate plays a large role elsewhere in their work as well.

The technique of conformal mapping then allows us to obtain a more detailed rigidity theorem on a fixed-size neighborhood of each hairpin-like structure.

Theorem 1.3. *There are absolute constants r_0, κ_0, and N_0 such that if u is a classical solution to* (1) *in B_1 satisfying* (3)*, then there is N, $0 \le N \le N_0$ and N saddle points $\{z_j\}_{j=1}^N$ of u with the following properties:*

(a) $F(u)$ has curvature at most κ_0 on $F(u) \cap B_{1/2} \setminus \bigcup_{j=1}^{N} B_{r_0}(z_j)$.

(b) For each j, $a_j := u(z_j) \le r_0/100$, and there is an injective conformal mapping

$$\phi_j : B_{2r_0} \cap \bar{\Omega}_{a_j} \to \mathbb{R}^2 \quad \text{such that} \quad \phi_j(0) = z_j, \quad \text{and} \quad B_{r_0}(z_j)^+(u) \subset \phi_j(B_{2r_0} \cap \Omega_{a_j}) \subset B_{4r_0}(z_j)^+(u).$$

Moreover, there is $\theta_j \in \mathbb{R}$ such that for all $z \in B_{2r_0} \cap \Omega_{a_j}$,

$$|\phi_j'(z) - e^{i\theta_j}| \le |z|/(100r_0); \quad |\phi_j''(z)| \le 1/(100r_0).$$

(c) If κ denotes the curvature of $F(u)$ and κ_a denotes the curvature of $\partial\Omega_a$, then

$$|\kappa(\phi_j(z)) - \kappa_{a_j}(z)| \le 1/(100r_0), \quad z \in B_{2r_0} \cap \partial\Omega_{a_j}.$$

To interpret part (c) of this theorem, note that

$$\kappa_a(z) \sim a/|z|^2, \quad z \in \partial\Omega_a$$

Hence

$$|z| \le \sqrt{ar_0} \implies \kappa_a(z) \gg \frac{1}{100r_0}$$

Furthermore, a is comparable to the separation distance between the two hairpins. Thus, for points closer to z_j than the geometric mean of the separation distance between the two hairpins and the distance r_0, the bound in part (c) says that the curvature of the approximate hairpins is close to that of the standard model. In particular, the two components of the zero set are convex in this range. At distances significantly larger than this geometric mean, one can no longer guarantee that $\kappa(\phi_j(z))$ is positive, but the bound in part (c) still implies that $|\kappa(\phi_j(z)| \le 1/(50r_0)$. This is a nontrivial bound. At the largest scale, $r_0 < |z| < 2r_0$ it is the same as the standard interior 2nd derivative bounds for flat free boundaries, but at smaller dyadic scales it is a stronger curvature constraint.

In [Tra14], Traizet found a remarkable change of variables that converts the free boundary problem into a problem about minimal surfaces with a plane of symmetry. If $|\nabla u| < 1$, then the minimal surfaces are embedded, and otherwise they are immersed. This means that although neither problem is strictly contained in the other, there is direct overlap between the results of Colding and Minicozzi and the results proved here. The extra hypothesis $|\nabla u| < 1$ removes nearly all the difficulties from the free boundary classification problem we are considering because in that case the zero set of u consists of convex components. Nevertheless, in this simple overlapping case Traizet's change of variables allows us to make a direct comparison with results of [CM02].

Under Traizet's correspondence, the standard double hairpin becomes the standard catenoid,

$$\Sigma_\rho = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_2/\rho)^2 + (x_3/\rho)^2 = \cosh^2(x_1/\rho))\}, \quad \rho > 0.$$

Denote $\mathcal{B}_r := \{x \in \mathbb{R}^3 : |x| < r\}$.

Corollary 1.4. *Let $M \subseteq \mathcal{B}_R$ be an embedded minimal surface, homeomorphic to an annulus, with $\partial M \subseteq \partial\mathcal{B}_R$. Suppose that M is symmetric with respect to the reflection $x_3 \mapsto -x_3$ and that $M^+ = M \cap \{x_3 > 0\}$ is a simply-connected graph over the x_1x_2-plane. Suppose that the shortest closed geodesic of M has length ϵ and passes through the origin in $\mathcal{B}_R$. There are absolute constants $R_0 < \infty$ and $\epsilon_0 > 0$ such that if $R \ge R_0$ and $\epsilon \le \epsilon_0$, then there exists $\rho > 0$,*

$$|2\pi\rho - \epsilon| \le \epsilon/100$$

and an injective conformal mapping $\phi : \Sigma_\rho \cap \mathcal{B}_1 \to M$ that is isometric up to a factor $1 \pm |x|/100$, and the Gauss curvatures K of M and K_ρ of Σ_ρ are related by

$$|K(\phi(x)) - K_\rho(x)| \le \begin{cases} (1/100)(\epsilon/|x|^2), & |x| \le \sqrt{\epsilon} \\ 1/100, & \sqrt{\epsilon} \le |x| \le 1 \end{cases}$$

Note that because $K_\rho(x) \sim -\rho^2/|x|^4$ and $\epsilon \approx \rho$, in the range $|x| < \sqrt{\epsilon}$, the curvatures are close. This is the same bound as (but in much less generality than) the sharpest result of Colding and Minicozzi (see [CM02, Remark 3.8]). On the other hand, our corollary gives nontrivial rigidity for both distance distortion and curvature in the range $\sqrt{\epsilon} \ll |x| \ll 1$. This range is not addressed in [CM02], and the present result suggests that there may be interpolating rigidity estimates all the way to unit scale that are valid in the case of general embedded minimal annuli.

1.1. **Outline of the paper.** The first seven sections of the paper are devoted to the proof of Theorem 1.1. In Section 3 we establish the universal Lipschitz and *nondegeneracy* bounds enjoyed by the sequence of solutions u_k. Section 4 describes the two notions of weak solutions – viscosity and variational – that are preserved under the limit. In Section 5 we recall the Weiss Monotonicity Formula [Wei98] and use it to characterize the blow-up/blow-down limit of a weak nondegenerate solution; there are two possibilities (up to rigid motion): the half-plane solution $P(x) = x_2^+$ or

$$V(x) = s|x_2| \quad \text{for some } 0 < s \leq 1.$$

Weak solutions approaching the half-plane solution are well understood by the classical results of Caffarelli [Caf87, Caf89] and our focus will be to understand the structure of classical solutions that are close to V. The first step is carried out in Section 6, where we prove some auxiliary lemmas concerning the structure of their free boundary. We also establish the key fact that the gradient magnitude of weak solutions, which blow down to V, is bounded above by 1; this, in turn, translates to the strong geometric property that $F(u)$ has non-negative curvature wherever it's smooth. The latter will be a key element in the proof of Theorem 1.1, carried out in Section 7.

In Section 8 we start exploring the local structure of a solution u, satisfying (3), in the unit disk B_1. We delineate a dichotomy – if near a point p of the free boundary there are two connected components of the zero phase close enough to each other at a distance $O(a)$, then u resembles $|x_2|$ (up to a rigid motion) in a unit-size neighborhood $B_{r_0}(p)$ (this scenario will ultimately lead to u resembling a hairpin solution); otherwise, the free boundary has bounded curvature at p. Sections 9 and 10 are devoted to exploring the first branch of the dichotomy. In Section 9 we show that that the free boundary from scale r_0 all the way down to scale $O(a)$ consists of four curves that have bounded turning in the outer scales. In the penultimate Section 10 we finally see the hairpin arising in the inner scale and we systematically treat both scales by constructing an injective holomorphic map (Lemma 10.5) from the positive phase of u in $B_{r_0}(p)$ to the positive phase of an appropriate hairpin solution H_a. Obtaining estimates on the second derivative of the map in Lemma 10.6 allows us to relate the curvature of $F(u)$ to the curvature of $F(H_a)$ of a model hairpin solution.

In the last Section 11 we exploit the Traizet correspondence to prove Corollary 1.4.

2. Notation.

The disk of radius r centered at $x = (x_1, x_2) \in \mathbb{R}^2$ will be denoted by $B_r(x)$. When the argument is absent, we are referring to the disk centered at the origin, $B_r := B_r(0)$. The unit vectors along x_1 and x_2 will be denoted by e_1 and e_2, respectively. The three-dimensional ball of radius r, centered at $p \in \mathbb{R}^3$, will be denoted by $\mathcal{B}_r(p)$.

If Ω is an open set of $\mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ is a non-negative function, define the *positive phase* of u to be

$$\Omega^+(u) := \{x \in \Omega : u(x) > 0\}$$

and its *free boundary* $F(u) := \partial\Omega^+(u) \cap \Omega$.

If $S \subseteq \mathbb{R}^2$, a δ-neighborhood of S will be denoted by

$$\mathcal{N}_\delta(S) := \bigcup_{x \in S} B_\delta(x).$$

Denote the distance between two non-empty sets U, V by

$$d(U, V) = \inf\{|p - q| : p \in U, q \in V\},$$

while the Hausdorff distance between two compact subsets K_1, K_2 of $\mathbb{R}^2$ will be denoted by

$$d_H(K_1, K_2) = \inf\{\delta > 0 : K_1 \subseteq \mathcal{N}_\delta(K_2) \text{ and } K_2 \subseteq \mathcal{N}_\delta(K_1)\}.$$

By $\mathcal{H}^1$ we shall refer to the one-dimensional Hausdorff measure.

In all that follows $C, c, c', \tilde{c}, c_0, c_1, c_2$, etc. will denote positive numerical constants. The constants in the O-notation, wherever used, are also meant to be numerical.

3. Preliminaries.

Let u be a solution of (1) in a disk $B \subseteq \mathbb{R}^2$ that satisfies (3). In our forthcoming arguments we shall often be working with some connected component U of $[B_r(x)]^+(u)$, where $B_r(x) \Subset K \Subset B$ for some compact set K. Claim that U is a piecewise smooth domain; that will provide us with enough regularity to apply the Divergence Theorem in U. It suffices to show that only finitely many connected components of $F(u)$ intersect $\partial B_r(x)$ and that each intersects it only a finite number of times. Let γ be any connected component of $F(u)$ intersecting K. Since for each $p \in F(u)$, $F(u) \cap B_{\epsilon(p)}(p)$ is locally the graph of a smooth function when $\epsilon(p)$ is small enough, the compact $\gamma \cap K$ has a finite subcover $\{B_{\epsilon(p_i)}(p_i), p_i \in \gamma \cap K\}_{i=1}^N$, so that

$$d\left(\gamma \cap K, (F(u) \setminus \gamma) \cap K\right) \geq \delta(\gamma) := \frac{1}{2} \min\{\epsilon_{p_i}\}_{i=1}^N. \tag{4}$$

But $\{\mathcal{N}_{\delta(\gamma)}(\gamma \cap K)\}_\gamma$, where γ ranges over all connected curves of $F(u)$ intersecting K, is a cover of the compact $F(u) \cap K$, so it has a finite subcover $\{\mathcal{N}_{\delta(\gamma_j)}(\gamma_j \cap K)\}_{j=1}^M$. Because of (4), each element of the subcover contains only $\gamma_j \cap K$ and nothing else from $F(u) \cap K$, so there are only finitely many curves γ intersecting K and thus $B_r(x)$. Each such γ intersects $\partial B_r(x)$ only a finite number of times, because by the classical result of [KN77], the free boundary $F(u)$ is real analytic.

We shall now prove two fundamental regularity properties that classical solutions of (1) given (3) satisfy: universal Lipschitz bound and universal non-degeneracy away from the free boundary. To elucidate the latter part of our claim, let us state the relevant definition.

Definition 3.1. *A non-negative function $u : \Omega \to \mathbb{R}$ is **non-degenerate** if there exists a constant $c > 0$, such that*

$$\sup_{B_r(x)} u \geq cr$$

for every $B_r(x) \subseteq \Omega$ centered at a point $x_0 \in F(u)$.

First, let us show that classical solutions enjoy a universal Lipschitz bound.

Proposition 3.1 (Lipschitz bound)**.** *Let u be a classical solution of* (1) *in $B_R(0)$. If the largest disk in $B_R^+(u)$, centered at x, touches $F(u)$, then*

$$|\nabla u|(x) \leq C.$$

for some numerical constant $C > 0$. In particular, if $0 \in F(u)$

$$\|\nabla u\|_{L^\infty(B_{R/2})} \leq C. \tag{5}$$

Proof. If $u(x) = m$, then by Harnack's inequality $c_1 m \leq u(y) \leq c_2 m$ on $\partial B_{r/2}(x)$. Let h be the harmonic function in the annulus $A_r(x) := B_r(x) \setminus B_{r/2}(x)$, whose boundary values are:

$$\begin{aligned} h &= c_1 m \quad \text{on} \quad \partial B_{r/2}(x) \\ h &= 0 \quad \text{on} \quad \partial B_r(x). \end{aligned}$$

By the maximum principle $h \leq u$ in A_r and so by the Hopf lemma,

$$h_\nu(p) \leq u_\nu(p) = 1,$$

where ν denotes the inner-normal to B_R^+ and $p \in F(u)$ is a point of touching between $F(u)$ and $B_r(x)$. On the other hand, $h_\nu(p) \geq c'm/r$, thus

$$m \leq C'r.$$

Thus,

$$|\nabla u|(x) \leq \frac{c_0}{r} \fint_{\partial B_{r/2}} u \, d\mathcal{H}^1 \leq \frac{c_0 c_2 m}{r} \leq C,$$

for some numerical constant C.

Statement (5) follows once we point out that for $x \in B_{R/2}$ the largest ball contained in $B_R^+(u)$ and centered at x, will certainly touch $F(u)$. $\square$

The universal nondegeneracy property is established through the following proposition.

Proposition 3.2. *Let u be a classical solution of* (1) *in $B_R(0)$, for which* (3) *is satisfied. Assume further that $0 \in F(u)$. Then*

$$\sup_{B_r(0)} u = \max_{\partial B_r(0)} u \geq \frac{1}{2\pi} r \quad \text{for all} \quad 0 < r < R.$$

Proof. Since u is continuous and subharmonic, the maximum principle implies $\sup_{B_r(0)} u = \max_{\partial B_r(0)} u$. Let $\tilde{u}(x) := r^{-1}u(rx)$ denote the r-rescale of u. It suffices to show that $\sup_{\partial B_1} \tilde{u} \geq 1/2\pi$.

Let $\phi : [0,1] \to \mathbb{R}$ be the function

$$\phi(t) = \begin{cases} \frac{1}{2} & 0 \leq t \leq \frac{1}{2} \\ 1 - t & \frac{1}{2} < t \leq 1 \end{cases}.$$

and let $\psi(x) = \phi(|x|)$. Let U be the component of $B^+_{R/r}(\tilde{u}) = r^{-1}B^+_R(u)$ in B_1 whose boundary contains the origin. Then if $\tilde{u}_\nu$ denotes the inner normal to U,

$$-\int_U \nabla\psi \cdot \nabla\tilde{u}\, dx = -\int_U \operatorname{div}(\psi\nabla\tilde{u})\, dx = \int_{\partial U \cap B_1} \psi\tilde{u}_\nu\, d\mathcal{H}^1 = \int_{\partial U \cap B_1} \psi\, d\mathcal{H}^1.$$

On the other hand, if $\hat{r}$ denotes the unit vector field in the radial direction,

$$\begin{aligned} -\int_U \nabla\psi \cdot \nabla\tilde{u}\, dx &= \int_{U \setminus B_{1/2}} \operatorname{div}(\tilde{u}\hat{r}) - \operatorname{div}(\hat{r})\tilde{u}\, dx = \\ &= \int_{\partial B_1 \cap U} \tilde{u}\, d\mathcal{H}^1 - \int_{\partial B_{1/2} \cap U} \tilde{u}\, d\mathcal{H}^1 - \int_{U \setminus B_{1/2}} \frac{\tilde{u}}{|x|}\, dx. \end{aligned}$$

Therefore, as $\mathcal{H}^1(\partial U \cap B_1) \geq 2$,

$$\int_{\partial B_1 \cap U} \tilde{u}\, d\mathcal{H}^1 \geq \int_{F(\tilde{u}) \cap U} \psi\, d\mathcal{H}^1 \geq \frac{1}{2}\mathcal{H}^1(\partial U \cap B_1) \geq 1.$$

Hence, $\sup_{\partial B_1 \cap U} \tilde{u} \geq 1/2\pi$. □

4. Weak solutions.

In this section we define the two notions of weak solutions that will be useful in classifying the limits of sequences of classical solutions. Let

$$I[u,\Omega] = \int_\Omega |\nabla u|^2 + 1_{\{u>0\}}\, dx \quad \Omega \subseteq \mathbb{R}^2$$

be the one-phase energy functional whose Euler-Lagrange equation is the free boundary problem (1).

Definition 4.1. *The function $u \in H^1_{loc}(\Omega)$ is a **variational solution** of* (1) *if $u \in C(\Omega) \cap C^2(\Omega^+(u))$ and*

$$0 = L[u](\phi) := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} I[u(x + \epsilon\phi(x))] = \int_\Omega \left(|\nabla u|^2 + 1_{\{u>0\}}\right) div\ \phi - 2\nabla u D\phi(\nabla u)^T\, dx$$

for any $\phi \in C^\infty_c(\Omega; \mathbb{R}^2)$.

The next proposition is standard and says that any globally defined limit of uniformly convergent variational solutions that are uniformly Lipschitz continuous and uniformly non-degenerate, inherits the same properties.

Proposition 4.1. *Let $\{u_k\} \in H^1_{loc}(B_{R_k})$, $R_k \nearrow \infty$, be a sequence of variational solutions of* (1) *which satisfies*

- *(Uniform Lischitz continuity) There exists a constant C, such that $\|\nabla u_k\|_{L^\infty(B_{R_k})} \leq C$;*
- *(Uniform non-degeneracy) There exists a constant c, such that $\sup_{B_r(x)} u_k \geq cr$ for every $B_r(x) \subseteq B_{R_k}$, centered at a free boundary point $x \in F(u_k)$.*

Then any limit $u \in H^1_{loc}(\mathbb{R}^2)$ of a uniformly convergent on compacts subsequence $u_k \to u$ satisfies

(a) $\overline{\{u_k > 0\}} \to \overline{\{u > 0\}}$ and $F(u_k) \to F(u)$ locally in the Hausdorff distance;
(b) $1_{\{u_k>0\}} \to 1_{\{u>0\}}$ in $L^1_{loc}(\mathbb{R}^2)$;
(c) $\nabla u_k \to \nabla u$ a.e.

Moreover, u is a Lipschitz continuous, non-degenerate variational solution of (1).

Proof. Obviously, u is a global Lipschitz continuous function with $\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq C$ and $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$. One proves properties a) through c) arguing as in [CS05, Lemma 1.21]. The non-degeneracy of u follows from the non-degeneracy of u_k combined with the fact that $F(u_k) \to F(u)$ locally in the Hausdorff distance.

To show that u is a variational solution as well, note that since $\nabla u_k \to \nabla u$ a.e. and $|\nabla u_k|$ and $|\nabla u|$ are bounded above by C, the Dominated Convergence Theorem implies that for every $\phi \in C^1_c(\mathbb{R}^2;\mathbb{R}^2)$

$$0 = \lim_{k\to\infty} L[u_k](\phi) = L[u](\phi).$$

□

The second notion of weak solution that will make use of is that of a viscosity super/sub-solution ([CS05]).

Definition 4.2. *A viscosity supersolution (resp. subsolution) of* (1) *is a non-negative continuous function w in Ω such that*

- $\Delta w \leq 0$ *(resp. $\Delta w \geq 0$) in $\Omega^+(w)$;*
- *If $x_0 \in F(w)$ and there is a disk $B \subseteq \Omega^+(w)$ (resp. $B \subseteq \{w = 0\}$) that touches $F(w)$ at x_0, then near x_0 in B (resp. B^c), in every non-tangential region,*

$$w(x) = \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \quad \text{for some} \quad \alpha \leq 1 \ (\text{resp. } \alpha \geq 1),$$

where ν denotes the inner (resp. outer) unit normal to ∂B at x_0.

A function w is a ***viscosity solution*** *if w is both a viscosity super- and subsolution.*

The class of viscosity solutions is well-suited for taking uniform limits in compact sets.

Lemma 4.2 (Limit of viscosity solutions)**.** *Let $u_k \in C(\Omega)$ be a sequence of viscosity solutions of* (1) *in Ω such that $u_k \to u$ uniformly and u is Lipschitz continuous. Then u is also a viscosity supersolution of* (1) *in Ω. If, in addition, $\overline{\Omega^+(u_k)} \to \overline{\Omega^+(u)}$ locally in the Hausdorff distance, then u is a viscosity subsolution, as well.*

Proof. Clearly $\Delta u = 0$ in $\Omega^+(u)$, so we only need to check that the appropriate free boundary conditions are satisfied.

Let us show that u satisfies the viscosity supersolution condition at the free boundary. Assume there is a disk B touching $x_0 \in F(u)$ from the positive phase. Without loss of generality, $x_0 = 0$ and the unit normal of ∂B at 0 is $\nu = e_2$. According to [CS05, Lemma 11.17], u has the linear behaviour:

$$u(x) = \alpha x_2 + o(|x|) \quad \text{in non-tangential regions of } B$$

for some $0 < \alpha < \infty$ where ν denotes the inner unit normal to ∂B at x_0. Claim that $\alpha \leq 1$. Fix $\epsilon > 0$ small. If we blow up at 0,

$$u_\lambda(x) := \lambda^{-1} u(x_0 + \lambda x) \to \alpha x_2 \quad \text{in } B_1 \cap \{x_2 > \epsilon\} \quad \text{uniformly as } \lambda \to 0.$$

Denote $(u_k)_\lambda(x) = \lambda^{-1}u(\lambda x)$ the dilate of u_k at 0. By the uniform convergence of u_k to u, for some fixed small enough $\lambda > 0$

$$|(u_k)_\lambda(x) - \alpha x_2| < \alpha\epsilon/2 \quad \text{in } B_1 \cap \{x_2 > \epsilon\} \quad \text{for all large enough } k. \tag{6}$$

Consider the perturbation D_t of the domain $B_1 \cap \{x_2 > \epsilon\}$ defined by

$$D_t = \{x \in B_1 : x_2 > \epsilon - t\eta(x_1)\},$$

where $0 \leq \eta(x_1) \leq 1$ is a smooth bump function supported in $|x_1| < 1/2$ with $\eta(x_1) = 1$ for $|x_1| \leq 1/4$. We know that $D_0 \Subset \Omega^+((u_k)_\lambda)$ and since $0 \in F(u_\lambda)$

$$F((u_k)_\lambda) \cap B_\epsilon \neq \emptyset. \tag{7}$$

for all large enough k. Pick a k such that both (6) and (7) hold. Then for some $0 < t_0 < 2\epsilon$ the domain $D_{t_0} \subseteq \Omega^+((u_k)_\lambda)$ will touch $F((u_k)_\lambda)$ at some $p \in F((u_k)_\lambda) \cap \{|x_1| < 1/2\}$. Define a harmonic function v in D_{t_0} with boundary values

$$v(x) = \begin{cases} \alpha x_2 - \alpha\epsilon & \text{on} \quad \partial B_1 \cap \{x_2 > \epsilon\} \\ 0 & \text{on} \quad B_1 \cap \{x_2 = \epsilon - t_0\eta(x_1)\} \end{cases}$$

Thus, by the maximum principle $v \leq (u_k)_\lambda$ in D_{t_0}, so that near p in non-tangential regions of D_{t_0}, for some $\tilde{\alpha} \leq 1$

$$v(x) \leq (u_k)_\lambda(x) = \tilde{\alpha}\langle x - p, \nu(p)\rangle + o(|x - x_0|),$$

where $\nu(p)$ is the inner normal to ∂D_{t_0} at p. On the other hand, a standard perturbation argument gives $v_\nu(p) = \alpha + O(\epsilon)$. Since ϵ is arbitrary, we conclude $\alpha \leq 1$.

Let us now assume that $\overline{\Omega^+(u_k)} \to \overline{\Omega^+(u)}$ in the Hausdorff distance and show that u satisfies the viscosity subsolution condition at the free boundary. Let there be a disk B touching $F(u)$ at x_0 from the zero phase. Without loss of generality, $x_0 = 0$ and the unit outer normal at ∂B is e_2. According to [CS05, Lemma 11.17], for some $0 \leq \beta < \infty$

$$u(x) \leq \beta x_2^+ + o(|x|).$$

Given $\epsilon > 0$ we can dilate u and u_k near 0 sufficiently, so that

$$(u_k)_\lambda(x) \leq u_\lambda(x) + \epsilon/2 \leq \beta x_2^+ + \epsilon \quad \text{in } B_1$$

for some fixed large λ and all large enough k. Moreover, since $\Omega^+((u_k)_\lambda) \to \Omega^+(u_\lambda)$, we can choose k large enough such that

$$\Omega^+((u_k)_\lambda) \cap B_1 \Subset \{x_2 > -\epsilon/2\} \quad \text{and} \quad F((u_k)_\lambda) \cap B_{\epsilon/2} \neq \emptyset.$$

Let E_t be the domain

$$E_t = \{x \in B_1 : x_2 > -\epsilon + t\eta(x_1)\}$$

and note that for some $0 < t_0 < 2\epsilon$, $E_{t_0} \supseteq \Omega^+((u_k)_\lambda) \cap B_1$ and ∂E_{t_0} touches $F((u_k)_\lambda) \cap B_1$ at some point $q \in F((u_k)_\lambda \cap \{|x_1| < 1/2\}$. Define a harmonic function w in E_{t_0} having boundary values:

$$w(x) = \begin{cases} \beta x_2^+ + \min\big((2(x_2+\epsilon)^+), \epsilon\big) & \text{on} \quad \partial B_1 \cap \{x_2 > -\epsilon\} \\ 0 & \text{on} \quad B_1 \cap \{x_2 = -\epsilon + t_0\eta(x_1)\} \end{cases}$$

Thus, the maximum principle implies that near q, in non-tangential regions of $\Omega^+((u_k)_\lambda)$,

$$w(x) \geq (u_k)_\lambda(x) = \tilde{\beta}\langle x - q, \nu(q)\rangle + o(|x - x_0|),$$

for some $\tilde{\beta} \geq 1$. Hence, $w_\nu(q) \geq \tilde{\beta} \geq 1$. On the other hand, a standard perturbation argument gives $w_\nu(q) = \beta + O(\epsilon)$. Since ϵ is arbitrary, we conclude that $\beta \geq 1$. $\square$

5. Characterization of blow-downs and blow-ups.

The notion of a variational solution is incredibly useful precisely because it admits the application of the powerful Weiss Monotonicity Formula.

Lemma 5.1 (Weiss' Monotonicity Formula, Theorem 3.1 in [Wei98])**.** *Let u be a variational solution of* (1) *in $\Omega \subseteq \mathbb{R}^n$ and that $B_R(x_0) \subseteq \Omega$. Then*

$$\Phi(u, r) := r^{-n} \int_{B_r(x_0)} \left(|\nabla u|^2 + 1_{\{u>0\}}\right)\, dx - r^{-n-1} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1} \tag{8}$$

satisfies the monotonicity formula

$$\Phi(u, r_2) - \Phi(u, r_1) = \int_{B_{r_2}(x_0)\setminus B_{r_1}(x_0)} 2|x|^{-n-2}\big(\nabla u \cdot (x - x_0) - u\big)^2 \, dx \geq 0 \tag{9}$$

for $0 < r_1 < r_2 < R$.

Lemma 5.2. *Let u be a variational solution of* (1) *in $\mathbb{R}^n$ which is globally Lipschitz. Assume $0 \in F(u)$ and let v be any limit of a uniformly convergent on compacts subsequence of*

$$v_j(x) = R_j^{-1} u(R_j x)$$

as $R_j \to \infty$. Then v is Lipschitz continuous and homogeneous of degree one.

Proof. Denote $v_j(x) = R_j^{-1}u(R_jx)$ and note that v_j are also global variational solutions of (1) and $\Phi(v_j, r) = \Phi(u, rR_j)$. According to Lemma 5.1 the quantity $\Phi(u, R)$ is non-decreasing as $R \to \infty$ and, moreover, it is uniformly bounded since u is Lipshitz continuous. Hence, for any fixed $0 < r_1 < r_2$

$$0 = \lim_{j\to\infty} \big(\Phi(u, r_2R_j) - \Phi(u, r_1R_j)\big) = \lim_{j\to\infty} \big(\Phi(v_j, r_2) - \Phi(v_j, r_1)\big)$$

and (9) yields

$$\lim_{j\to\infty} \int_{B_{r_2}\setminus B_{r_1}} 2|x|^{-n-2}\big(\nabla v_j \cdot x - v_j\big)^2 \, dx = 0.$$

Possibly passing to a subsequence such that $\nabla v_j \rightharpoonup \nabla v$ weakly in L^2, the lower semicontinuity of the L^2-norm with respect to weak convergence implies

$$\int_{B_{r_2}\setminus B_{r_1}} 2|x|^{-n-2}\big(\nabla v \cdot x - v\big)^2 \, dx = 0.$$

Thus, $\nabla v \cdot x = v$ a.e. whence it is a standard exercise to conclude that v is homogeneous of degree one. □

Proposition 5.3 (Characterization of blowdowns)**.** *Let u be both a viscosity and a variational solution of (1) in $\mathbb{R}^2$, which is Lipschitz-continuous and non-degenerate. Assume $0 \in F(u)$ and let v be any limit of a uniformly convergent on compacts subsequence of*

$$v_j(x) = R_j^{-1}u(R_jx)$$

as $R_j \to \infty$. Then v is either $V_1(x) = x_2^+$ or $V_2(x) = s|x_2|$ for some $0 < s \le 1$ in an appropriately chosen Euclidean coordinate system.

Proof. As a consequence of Proposition 4.1, Lemma 4.2 and Lemma 5.2 applied to the sequence v_j we conclude that v is a Lipschitz continuous, non-degenerate, viscosity and variational solution of (1), which is homogeneous of degree 1. Thus, after possibly rotating the coordinate axes

$$v(x) = c_1x_2^+ + c_2x_2^-,$$

where $c_1 \ge c_2 \ge 0$. We have the following two cases.
Case 1 ($c_2 = 0$). By non-degeneracy we must have $c_1 > 0$ and since every point $x_0 \in F(v) = \{x_2 = 0\}$ has a tangent disk from both the positive and zero set of v, then $c_1 = 1$.
Case 2 ($c_2 > 0$). Every point $x_0 \in F(v) = \{x_2 = 0\}$ has a tangent disk from the positive phase of v only, so from the fact that v is a viscosity solution we can just conclude that $1 \ge c_1 \ge c_2$. On the other hand, v is also a variational solution and an easy computation gives

$$0 = L[v](\phi) = (c_1^2 - c_2^2)\int_{\mathbb{R}} \phi_2(x_1, 0)dx_1$$

for any $\phi = (\phi_1, \phi_2) \in C_c^1(\mathbb{R}^2; \mathbb{R}^2)$. Thus, $c_1 = c_2 = s$. □

Exactly analogous arguments apply to *blow-up* limits of Lipschitz continuous weak solutions, so we have the analogous characterization:

Proposition 5.4 (Characterization of blow-ups)**.** *Let u be a both a viscosity and a variational solution of (1) in $\Omega \subseteq \mathbb{R}^2$, which is Lipschitz-continuous and non-degenerate. Assume $0 \in F(u)$ and let $v : \mathbb{R}^2 \to \mathbb{R}$ be any limit of a uniformly convergent on compacts subsequence of*

$$v_j(x) = \epsilon_j^{-1}u(\epsilon_jx)$$

as $\epsilon_j \to 0$. Then v is either $V_1(x) = x_2^+$ or $V_2(x) = s|x_2|$ for some $0 < s \le 1$ in an appropriately chosen Euclidean coordinate system.

6. Auxiliary Lemmas.

Lemma 6.1. *Let u be a classical solution of (1) in a domain B_2, which has Lipschitz norm L and such that*

$$|u(x) - s|x_2|| < \epsilon \quad in \quad B_2 \tag{10}$$

for some $0 < s \leq 1$ and some small $\epsilon > 0$. Then there exists a universal constant $c > 0$ such that

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq 1 + cL\sqrt{\epsilon}.$$

Proof. Assumption (10) implies $B_2^+(u_R) \subset \{|x_2| > \epsilon/s\}$. Thus, at any $p \in \partial B_1 \cap \{|x_2| > 2M\epsilon/s\}\}$ for a large $M \leq s/2\epsilon$, we have $B_{M\epsilon/s}(p) \subset B_2^+(u)$ so that $u - s|x_2|$ is harmonic in $B_{M\epsilon/s}(p)$. Hence,

$$|\nabla u(p) - s\nabla|x_2|(p)| \leq \frac{c'}{M\epsilon/s} \fint_{\partial B_{M\epsilon/s}} \big|u - s|x_2|\big| \, d\mathcal{H}^1 \leq \frac{c'}{M/s},$$

which in turn leads to

$$|\nabla u(p)|^2 \leq \left(s + \frac{c'}{M/s}\right)^2 \leq 1 + \frac{3c'}{M/s}. \tag{11}$$

Define the function $v : B_2 \to \mathbb{R}$

$$v = \begin{cases} |\nabla u|^2 - 1 & \text{in } B_2^+(u) \\ 0 & \text{otherwise.} \end{cases}$$

Then v is continuous in B_2 and since

$$\Delta|\nabla u|^2 = 2|D^2u|^2 + 2\nabla(\Delta u) \cdot \nabla u = 2|D^2u|^2 \geq 0 \quad \text{in } B_2^+(u),$$

v is subharmonic in $B_2^+(u)$. Let $v_h : \overline{B_1} \to \mathbb{R}$ be the harmonic function whose boundary values on ∂B_1 are given by

$$v_h(x) = \max\{v(x), 3c's/M\} \quad x \in \partial B_1.$$

By the maximum principle, $v_h > 0$ in B_1 and $v_h \geq v$ in $B_1^+(u)$, whence $v \leq v_h$ in B_1. By Poisson's formula, for any $x \in B_{1/2}$

$$\begin{aligned} v_h(x) \leq c\int_{\partial B_1} v_h \, d\mathcal{H}^1 &= c\left(\int_{\partial B_1 \cap \{|x_2| \leq M\epsilon/s\}} v_h \, d\mathcal{H}^1 + \int_{\partial B_1 \cap \{|x_2| > M\epsilon/s\}} v_h \, d\mathcal{H}^1\right) \\ &\leq \tilde{c}L^2\epsilon M/s + \frac{\tilde{c}}{M/s}, \end{aligned} \tag{12}$$

where the last inequality is a consequence of (11) and the Lipschitz control of u. Choosing $M = s/(\sqrt{\epsilon}L\})$ yields

$$v \leq v_h \leq 2\tilde{c}L\sqrt{\epsilon} \quad \text{in } B_{1/2}$$

which is the desired estimate. □

Lemma 6.2. *Let u_k be a sequence of classical solutions of (1) in B_{R_k}, $R_k \nearrow \infty$ that are uniformly Lischitz continuous and assume the sequence converges uniformly on compact subsets of $\mathbb{R}^2$ to $u : \mathbb{R}^2 \to \mathbb{R}$ with $0 \in F(u)$. If a blowdown limit of u*

$$u_{R_j}(x) = R_j^{-1}u(R_jx) \to s|x_2| \quad \textit{uniformly on compacts} \quad \textit{as } R_j \to \infty,$$

for some $0 < s \leq 1$, then

$$|\nabla u| \leq 1 \quad a.e.$$

Proof. Fix $\epsilon > 0$ and find j large enough so that

$$|u_{R_j} - s|x_2|| < \epsilon/2 \quad \text{in } B_2.$$

Then for all large enough k, such that $|u_{R_j} - (u_k)_{R_j}| < \epsilon/2$ in B_2, we have

$$|(u_k)_{R_j} - s|x_2|| < \epsilon,$$

so that Lemma 6.1 yields the estimate

$$\|\nabla u_k\|_{L^\infty(B_{R_j/2})} = \|\nabla(u_k)_{R_j}\|_{L^\infty(B_{1/2})} \leq 1 + C\sqrt{\epsilon},$$

where C is a bound on the Lipschitz norm of u_k. At every $x \in \{u > 0\}$, for $d = u(x)$, the disk $B_{d/2C}(x) \subseteq \{u > 0\}$ as well as $B_{d/2C}(x) \subseteq \{u_k > 0\}$ for all k large enough. Since $u_k \to u$ uniformly in $B_{d/2C}(x)$, where the functions are harmonic, we also get $\nabla u_k(x) \to \nabla u(x)$ as $k \to \infty$. Since $\nabla u(x) = 0$ a.e. $x \in \{u = 0\}$,

$$\|\nabla u\|_{L^\infty(B_{R_j/2})} = \lim_{k\to\infty} \|\nabla u_k\|_{L^\infty(B_{R_j/2})} \leq 1 + C\sqrt{\epsilon}.$$

Letting $R_j \to \infty$, followed by $\epsilon \to 0$, yields the result. □

Lemma 6.3. *Let u be a classical solution of* (1) *in $\Omega \subseteq \mathbb{R}^2$. Then the signed curvature κ of $F(u)$ is given by*

$$\kappa = -\frac{1}{2}\frac{\partial(|\nabla u|^2)}{\partial \nu},$$

where ν is the unit normal pointing towards $\Omega^+(u)$.

Proof. The curvature of a level set of a function v at a point where $|\nabla v| \neq 0$ is given by

$$\kappa = \operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right).$$

(Note that $\kappa > 0$ when the curvature vector points in the direction of $-\nabla v$, e.g. the curvature of the 0-level set of $v(x,y) = \log(x^2+y^2)$ is positive 1). Since in a small enough neighborhood U of each $x \in F(u)$ we can define a harmonic v which agrees with u on $U \cap \Omega^+(u)$, the curvature of $F(u)$ is given by the curvature of the $v = 0$ level set. Using in addition $|\nabla v|(x) = 1$, we compute

$$\kappa = \frac{\Delta v}{|\nabla v|} - \frac{\nabla v \cdot \nabla|\nabla v|}{|\nabla v|^2} = -\frac{(|\nabla v|^2)_\nu}{2|\nabla v|^2} = -(|\nabla u|^2)_\nu/2.$$

□

Remark 6.4. *If u is a classical solution of* (1) *in $\Omega \subseteq \mathbb{R}^2$ with $|\nabla u| < 1$ in $\Omega^+(u)$, then $F(u)$ has strictly positive curvature.*

Proof. The result follows immediately from Lemma 6.3 after an application of the Hopf Lemma to $|\nabla u|^2$, which is subharmonic in $\Omega^+(u)$:

$$\Delta|\nabla u|^2 = 2|D^2u|^2 + 2\nabla u \cdot \nabla(\Delta u) = 2|D^2 u|^2 \geq 0 \quad \text{in} \quad \Omega^+(u).$$

□

Lemma 6.5. *Let u be a classical solution of* (1) *in $\Omega \subset \mathbb{R}^2$, whose Lipschitz norm is $L < \infty$. If $V \subseteq \Omega^+(u)$, $V \Subset \Omega$ is a bounded, piecewise C^1 domain, then*

$$L^{-1}\mathcal{H}^1(\partial U \cap F(u)) \leq \mathcal{H}^1(\partial U \setminus F(u)).$$

Proof. Applying the Divergence Theorem in U:

$$0 = -\int_U \Delta u \, dx = \int_{\partial U \cap F(u)} u_\nu \, d\mathcal{H}^1 + \int_{\partial U \setminus F(u)} u_\nu \, d\mathcal{H}^1 = \mathcal{H}^1(\partial U \cap F(u)) + \int_{\partial U\setminus F(u)} u_\nu \, d\mathcal{H}^1,$$

where u_ν is the inner unit normal to ∂U. The result then follows from $|u_\nu| \leq L \quad \mathcal{H}^1$-a.e. on ∂U. □

Lemma 6.6. *Let u be a classical solution of* (1) *in B_3, which is Lipschitz continuous with norm L and for which assumption* (3) *is satisfied. There exists $\delta = \delta(L) > 0$ small enough such that if*

$$\{u = 0\} \subset B_3 \cap \{|x_2| < \delta\}$$

there are at most two connected components of $B_2^+(u)$ which intersect B_1, namely the connected component(s) containing $N = (0,1)$ and $S = (0,-1)$.

Proof. Consider a connected component U of $B_2^+(u)$ that contains neither N nor S; then it must be that $U \subseteq B_2 \cap \{|x_2| < \delta\}$. Assuming that U intersects B_1, by assumption (3) we have $\mathcal{H}^1(\partial U \cap F(u)) \geq 2$. On the other hand, U is a piecewise C^1 domain with $\partial U \setminus F(u) \subseteq \partial B_2 \cap \{|x_2| < \delta\}$, so that $\mathcal{H}^1(\partial U \setminus F(u)) \leq c\delta$ for some numerical constant $c > 0$. But then by Lemma 6.5

$$2/L \leq L^{-1}\mathcal{H}^1(\partial U \cap F(u)) \leq \mathcal{H}^1(\partial U \setminus F(u)) \leq c\delta,$$

which would be impossible if $\delta < 2/Lc$. □

Lemma 6.7. *Let u be a classical solution of* (1) *in B_4 for which assumption* (3) *is satisfied. Assume further that $(0,1)$ and $(0,-1)$ belong to two separate connected components of $B_2^+(u)$. Then there exists $\delta_0 > 0$ small enough such that if for any $0 < \delta \leq \delta_0$*

$$\{u = 0\} \subseteq \{|x_2| < \delta\},$$

the free boundary $F(u)$ inside $\{|x_1| < 1/2\}$ consists of two disjoint graphs:

$$F(u) \cap \{|x_1| < 1/2\} = \{x_2 = \phi_+(x_1) : |x_1| < 1/2\} \sqcup \{x_2 = \phi_-(x_1) : |x_1| < 1/2\},$$

for which $\phi_+ > \phi_-$ and

$$\|\phi_\pm\|_{C^{1,\alpha}(-1/2,1/2)} \leq C\delta$$

for some numerical positive constants C, $0 < \alpha < 1$.

Proof. By Proposition 3.1, $\|\nabla u\|_{L^\infty(B_2)} \leq L$ for some numerical constant L, so that by Lemma 6.6, there exists a small enough $\delta_0 > 0$ such that it is precisely the connected components U_+ and U_- of $B_2^+(u)$, containing $(0,1)$ and $(0,-1)$ respectively, that intersect B_1. Define the two functions u_+ and u_- on B_1 by $u_\pm = u 1_{U_\pm \cap B_1}$. Then each $u_\pm$ is a classical solution of (1) in B_1 whose free boundary $F(u_\pm)$ is contained in a flat strip $|x_2| < \delta$ with $u_+ = 0$ in $B_1 \cap \{x_2 < -\delta\}$ and $u_- = 0$ in $B_1 \cap \{x_2 > \delta\}$. By the classical result of Alt and Caffarelli [AC81], in $|x_1| < 1/2$ the free boundary $F(u_\pm)$ is the graph of a function $\phi_\pm : (-1/2, 1/2) \to \mathbb{R}$, which satisfies

$$\|\phi_\pm\|_{C^{1,\alpha}(-1/2,1/2)} \leq C\delta$$

for some $\alpha > 0, C > 0$. Noting that $F(u) \cap B_1 = F(u_+) \sqcup F(u_-)$, we are done. □

7. Characterization of the limit.

Recall the setup. We have a sequence $\{u_k\}$ of classical solutions of (1) in expanding disks B_{R_k}, $R_k \nearrow \infty$ with $0 \in F(u_k)$. Because of Proposition 3.1, u_k are uniformly Lipschitz on compact subsets of $\mathbb{R}^2$, so that up to a subsequence, u_k converges uniformly on compacts to some $u : \mathbb{R}^2 \to \mathbb{R}$. Moreover, since u_k are uniformly non-degenerate by Proposition 3.2, and, trivially also, weak solutions (variational and viscosity), then by Proposition 4.1 and Lemma 4.2, u is a global weak solution, which is Lipschitz continuous and non-degenerate. Thus, by Propositions 5.3 and 5.4, u blows down/blows up at a free boundary point to a half-plane or a wedge solution.

We shall show that, in for appropriately chosen Euclidean coordinates, u has to be one of the four:

- a *half-plane solution* $P(x) = x_2^+$
- a *two-plane solution* $W_b(x) = x_2^+ + (x_2 - b)^-$, for some $b < 0$
- a *wedge solution* $W_0(x) = |x_2|$
- a *hairpin solution* H_a, whose $\{H_a > 0\} = \{(x_1, x_2) : |ax_1 - (1 + \pi/2)| < \pi/2 + \cosh(ax_2)\}$ for some $a > 0$.

The proof of the classification Theorem 1.1 is realized in a sequence of propositions. Proposition 7.1 covers the scenario when u blows down to a half-plane solution, while Proposition 7.2 covers the case when the blowdown is a wedge solution. In the latter there is a dichotomy, u can be either a two-plane solution or satisfy $|\nabla u| < 1$ globally in its positive phase. The second scenario is the more subtle one and its treatment is carried out in several steps assembled under Proposition 7.3: in the first we employ a 2-point simultaneous blow-up to show that the free boundary is smooth everywhere but possibly one point; in the second step we prove the free boundary is actually smooth everywhere using the strong geometric constraint of positive free boundary curvature to argue that the zero phase contains a nontrivial sector with vertex at the exceptional point (cf. Lemmas 7.5 and 7.6); in the final step we establish that the free boundary must consist of exactly two smooth proper arcs, so that u has to be a hairpin solution, according to [Tra14, Theorem 12].

Let us commence with

Proposition 7.1. *Let u_k, u be as in Theorem 1.1 and assume $0 \in F(u)$. If a blowdown limit of u at 0 is a half-plane solution:*

$$R_j^{-1} u(R_j x) \to x_2^+ \quad as \quad R_j \nearrow \infty,$$

(with coordinates chosen appropriately), then $u = x_2^+$ itself.

Proof. Since $u_{R_j} \to x_2^+$ uniformly on compacts, the free boundary $F(u)$ is asymptotically flat, i.e.

$$F(u) \cap B_{R_j} \subseteq \{|x_2| \leq \epsilon_j\}$$

with the aspect ratio $\epsilon_j/R_j \to 0$ as $j \to \infty$. By the classical theorem of Alt and Caffarelli [AC81], $F(u)$ has to be the straight line $\{x_2 = 0\}$ and $u = x_2^+$. $\square$

Proposition 7.2. *Let u_k, u be as in Theorem 1.1 and assume $0 \in F(u)$ and that a blowdown limit of u at 0 is a wedge solution:*

$$R_j^{-1} u(R_j x) \to s|x_2| \quad as \quad R_j \nearrow \infty,$$

(with coordinates chosen appropriately) for some $0 < s \leq 1$. Then either $u = x_2^+ + (x_2 - b)^-$ for some $b \leq 0$, or

$$|\nabla u| < 1 \quad in \quad \{u > 0\}.$$

Proof. From Lemma 6.2 we have the bound $|\nabla u| \leq 1$ a.e. Noting that $|\nabla u|^2$ is a smooth subhamornic function in $\{u > 0\}$:

$$\Delta |\nabla u|^2/2 = 2|D^2 u|^2 \geq 0, \tag{13}$$

the Strong Maximum Principle entails that if $|\nabla u|^2(x_0) = 1$ at some $x_0 \in \{u > 0\}$, then $|\nabla u|^2 \equiv 1$ in the entire connected component U of x_0. Equation (13) implies that $|D^2 u|^2 = 0$ in U, so that u is an affine function in U. Thus U is a half-plane, say $U = \{x_2 < b\}$ for some $b \leq 0$ and $u = (x_2 - b)^-$ in U. The latter also implies that u has to blow down precisely to $|x_2|$, i.e. $s = 1$.

We shall now show that $v = u 1_{\mathbb{R}^2 \setminus U}$ is a viscosity solution itself. Once this is established, we can apply the previous Proposition 7.1 to v (as v has to blow down to x_2^+), so that v will have to be $v = x_2^+$ itself. We will then be able to conclude that $u = v + (x_2 - b)^- = x_2^+ + (x_2 - b)^-$.

Note that v is trivially a viscosity supersolution (as u is), so let us focus on showing that v is also a viscosity subsolution. Let $p \in F(v)$ and let $B \subseteq \{v = 0\}$ be a touching disk to $F(v)$ at p from the zero phase. If there exists a disk $B' \subseteq B$ such that $\partial B \cap \partial B' = p$ and $B' \subseteq \{u = 0\}$, then the subsolution condition for v will be inherited from u. Otherwise, every $B' \subseteq B$ with $\partial B \cap \partial B' = p$ will have to intersect the half-plane $U = \{x_2 < b\}$, and thus $B \subseteq U$ and $p \in \partial U$. But then, applying Proposition 5.4, we see that any blowup of u at p will have to equal $|(x - p) \cdot e_2|$, where e_2 is a unit vector in the direction of x_2. Hence, near p

$$v(x) = ((x - p) \cdot e_2)^+ + o(|x - p|) \quad \text{in any nontangential region of } B^c,$$

so that the subsolution condition is satisfied again.

$\square$

Proposition 7.3. *Let u_k, u be as in Theorem 1.1. Further assume that $|\nabla u| < 1$ in $\{u > 0\}$. Then u is a hairpin solution.*

Proof. We shall prove the proposition in three steps. In the first we show that $F(u)$ is smooth everywhere but possibly one point. In the second step we invoke Lemma 7.5 to establish that $F(u)$ is, in fact, smooth everywhere. In the final step we show that $F(u)$ consists of two disjoint proper arcs, so that by a result of Traizet [Tra14, Theorem 12] u has to be a hairpin solution.

Step 1. In order to establish the claim of the first step above, we have to prove that for no two distinct points $P_1, P_2 \in F(u)$ it can happen that u blows up to wedge solutions at both P_1 and P_2. From this it follows that at every point of $F(u)$ but possibly one, u has to blow up to the only other alternative – a half-plane solution (according to Proposition 5.4) so that $F(u)$ is smooth there.

Assume the contrary. Denote $u_\epsilon(x) := \epsilon^{-1} u(\epsilon x)$ and $(u_k)_\epsilon(x) := \epsilon^{-1} u_k(\epsilon x)$. If u blows up to wedge solutions at P_1 and P_2, Proposition 4.1a) implies there exist some unit vectors a_1 and a_2 such that given an arbitrary small $\lambda > 0$, one can find a sequence $\epsilon_j \searrow 0$ small enough such that for any $\epsilon = \epsilon_j$ small enough

$$d_H\left(\{u_\epsilon = 0\} \cap \overline{B_4}(P_i/\epsilon), \{P_i/\epsilon + t a_i : |t| \leq 4\}\right) < \lambda/2 \qquad i = 1, 2.$$

Further, for that particular fixed ϵ, Proposition 4.1a) implies that for all k large enough

$$d_H\left(\{(u_k)_\epsilon = 0\} \cap \overline{B_4}(P_i/\epsilon), \{u_\epsilon = 0\} \cap \overline{B_4}(P_i/\epsilon)\right) < \lambda/2 \qquad i = 1, 2.$$

As a consequence, for all k large enough

$$d_H\left(\{(u_k)_\epsilon = 0\} \cap \overline{B_4}(P_i/\epsilon), \{P_i/\epsilon + t a_i : |t| \leq 4\}\right) < \lambda \qquad i = 1, 2. \tag{14}$$

For ease of notation, denote $v := u_\epsilon$, $v_k := (u_k)_\epsilon$ and $Q_i = P_i/\epsilon$, $i = 1, 2$; let b_i be the vector a_i rotated by $\pi/2$. According to Lemma 6.6, there are at most two connected components of $B_2(Q_i)^+(v_k)$ that intersect $B_1(Q_i)$ – the one(s) that contain $N_i = Q_i + b_i$ and $S_i = Q_i - b_i$. We shall now show that there has to be just one if λ is small enough and k is large enough.

Assume N_1 and S_1 belong to two separate connected components $U_{+,k}$ and $U_{-,k}$ of $B_2(Q_1)^+(v_k)$. Combining this with (14) allows us to invoke Lemma 6.7 and conclude that $F(v_k)\cap\{|(x-Q_1)\cdot a_1| < 1/2\}$ consists of the graphs $\Sigma_{+,k}$ and $\Sigma_{-,k}$ of some functions $\phi_{+,k} > \phi_{-,k}$ over the line segment $I = \{Q_1 + ta_1 : |t| < 1/2\}$:

$$F(v_k) \cap \{|(x - Q_1) \cdot a_1| < 1/2\} = \Sigma_{+,k} \sqcup \Sigma_{-,k} \quad \text{where} \quad \Sigma_{\pm,k} = \{y + \phi_{\pm,k}(y)b_1 : y \in I\}.$$

Moreover, the functions $\phi_{\pm,k}$ satisfy the uniform bound

$$\|\phi_{\pm,k}\|_{C^{1,\alpha}(I)} \le C\lambda.$$

Hence, there exist $C^{1,\alpha}$ functions $\phi_\pm : I \to \mathbb{R}$ and a subsequence $\phi_{\pm,k_l}$ such that

$$\phi_{\pm,k_l} \to \phi_\pm \quad \text{in } C^1(I) \quad \text{as } l \to \infty.$$

But since $F(v_k) \to F(v)$ locally in the Hausdorff distance, it must be that $F(v) \cap \{|(x - Q_1) \cdot a_1| < 1/2\}$ consists precisely of the $C^{1,\alpha}$ graphs

$$\Sigma_\pm = \{y + \phi_\pm(y)b_1 : y \in I\}$$

and

$$B_2(Q_1)^+(v) \cap \{|(x - Q_1) \cdot a_1| < 1/2\} = U_+ \sqcup U_-,$$

where $U_+ = \{y + tb_1 : t > \phi_+(y) : y \in I\} \cap B_2(Q_1)$ and $U_- = \{y + tb_1 : t < \phi_-(y) : y \in I\} \cap B_2(Q_1)$. Moreover, since $v_k 1_{U_{\pm,k}}$ are viscosity solutions in $B_2(Q_1) \cap \{|(x - Q_1) \cdot a_1| < 1/2\}$ and $v_k 1_{(U_\pm)_k} \to v1_{U_\pm}$ uniformly there, Lemma 4.2 implies that $v1_{U_\pm}$ are viscosity solutions there as well. As their free boundary is $C^{1,\alpha}$, they are in fact classical solutions. But $|\nabla v| < 1$ in $\{v > 0\}$, so that according to Corollary 6.4, both Σ_+ and Σ_- have positive curvature. However, this is impossible, because $0 \in \Sigma_+ \cap \Sigma_-$, as v blows up at 0 to a wedge solution.

Hence N_1 and S_1 belong to the same connected component of $B_2(Q_1)^+(v_k)$ and similarly N_2 and S_2 belong to the same connected component of $B_2(Q_2)^+(v_k)$ for λ small enough and all k large enough.

Let

$$\alpha_L = \partial B_4(Q_1) \cap \{x : |(x - Q_1) \cdot b_1| < \lambda, (x - Q_1) \cdot a_1 < 0\}$$
$$\alpha_R = \partial B_4(Q_1) \cap \{x : |(x - Q_1) \cdot b_1| < \lambda, (x - Q_1) \cdot a_1 > 0\}.$$

By our topological assumption $F(v_k) \cap \overline{B_4}(Q_1)$ consists of arcs whose ends lie on $\alpha_L \cup \alpha_R$. Define F_L (resp. F_R) to be the set of points of $F(v_k) \cap \overline{B_4}(Q_1)$ that lie on arcs whose both ends are on α_L (resp. α_R). Then

$$F(v_k) \cap \overline{B_4}(Q_1) = F_L \sqcup F_R.$$

This is so, because the existence of an arc which has one end on α_L and the other on α_R would contradict the fact that N_1 from S_1 belong to the same connected component of $B_2(Q_1)^+(v_k)$.

Note that $F(v_k)\cap\overline{B_4}(Q_1)$ consists of a finite number of connected arcs. This follows from the analyticity of $F(v_k)$ which implies that only finitely many connected arcs of $F(v_k)$ can intersect $\partial B_4(Q_1)$, each intersecting it finitely many times. As a consequence, the sets F_L and F_R are closed and being bounded, they are compact. Hence, there exists a point $p \in F_L$ and a point $q \in F_R$ such that

$$|p - q| = \operatorname{dist}(F_L, F_R) < 2\lambda,$$

where the bound follows from (14). Denote by γ_L the arc of $F(v_k) \cap \overline{B_4}(Q_1)$ containing p, and by γ_R – the arc containing q.

Claim that the straight line segment $\tau_1 := \{(1 - t)p + tq : 0 < t < 1\}$, connecting p to q, lies in $B_4(Q_1)^+(v_k)$. Since p and q realize the distance between F_L and F_R, it must be that $\tau_1 \cap F(v_k) = \emptyset$, so that either $\tau_1 \subseteq B_4(Q_1)^+(v_k)$ or $\tau_1 \subseteq \{v_k = 0\} \cap B_4(Q_1)$. The latter alternative, however, is impossible, since $\gamma_L \cup \tau_1 \cup \gamma_R$ would disconnect N_1 from S_1 in $B_2(Q_1)^+(v_k)$.

Let us look globally at the connected arc(s) of $F(v_k)$ that contain p and q. One possibility is that p and q belong to the same arc. Let us argue that this is not the case. Denote by γ the arc of $F(v_k)$ with ends p

and q. Then $\gamma \cup \tau$ is a simple closed arc and it encloses a piecewise C^1 Jordan domain in the positive phase. Applying Lemma 6.5 to it, we see that, as $\mathcal{H}^1(\tau_1) < 2\lambda$,

$$2\lambda L > \mathcal{H}^1(\tau_1)L \geq \mathcal{H}^1(\gamma), \tag{15}$$

where L denotes the Lipschitz norm of v_k. However, since $\gamma_L \cup \tau_1 \cup \gamma_R \subseteq \gamma \sqcup \tau_1$ connects α_L to α_R,

$$\mathcal{H}^1(\gamma) + \mathcal{H}^1(\tau_1) \geq 2\sqrt{4^2 - \lambda^2},$$

so that $\mathcal{H}^1(\gamma) > 7$ for all λ small. Since L is bounded above by a universal constant, taking λ small enough leads to a contradiction in (15).

Thus, we may assume that p and q belong to distinct arcs of $F(v_k)$. We know that v blows down to a wedge solution (otherwise $|\nabla v(x)| = 1$ at some x) and without loss of generality, we may assume the blowdown is exactly $s|x_2|$. Thus, for any $\delta > 0$ there exists $M = M(\delta)$ large enough such that for all k large enough

$$\{v_k = 0\} \cap \overline{B_M} \subseteq \{|x_2| < \delta M\}.$$

Note that we may take M large enough so that both Q_1 and Q_2 belong to $B_{M/3}$ and $\tau_1 \subseteq B^+_{M/2}(v_k)$. Denote

$$\alpha_{L,M} = \partial B_M \cap \{x_1 < 0, |x_2| < \delta M\} \qquad \alpha_{R,M} = \partial B_M \cap \{x_1 > 0, |x_2| < \delta M\}.$$

and let γ_p be the arc of $F(v_k) \cap \overline{B_M}$ containing p, and γ_q – the arc of $F(v_k) \cap \overline{B_M}$ containing q. Let us show that γ_p and γ_q cannot both have an end on $\alpha_{L,M}$ (and, similarly, they cannot both have an end on $\alpha_{R,M}$). Assume they do: let $\tilde{\gamma}_p \subseteq \gamma_p$ be the subarc connecting $\alpha_{L,M}$ to p and $\tilde{\gamma}_q \subseteq \gamma_q$ be the subarc connecting $\alpha_{L,M}$ to q and let $\tilde{\alpha}$ be (smaller) circular arc on ∂B_M from the end $\partial B_M \cap \tilde{\gamma}_p$ to the end $\partial B_M \cap \tilde{\gamma}_q$. Then $\tilde{\alpha} \cup \tilde{\gamma}_p \cup \tilde{\gamma}_q \cup \tau_1$ encloses a Jordan domain $\tilde{O}$ and let $O \subseteq \tilde{O}^+(v_k)$ be the connected component of $\tilde{O}^+(v_k)$ whose boundary contains τ_1. Applying Lemma 6.5 to O, we quickly reach a contradiction for small δ, as

$$\mathcal{H}^1(F(v_k) \cap \partial O) \geq \mathcal{H}^1(\tilde{\gamma}_p) + \mathcal{H}^1(\tilde{\gamma}_q) \geq M/2 + M/2 = M$$

while

$$\mathcal{H}^1(\partial O \setminus F(v_k)) \leq \mathcal{H}^1(\tau_1) + \mathcal{H}^1(\tilde{\alpha}) < 2\lambda + c\delta M < 1 + c\delta M.$$

Therefore, it must be that γ_p has both its ends on $\alpha_{L,M}$ while γ_q has both its ends on $\alpha_{R,M}$. Of course, an analogous scenario holds near Q_2 as well: we can find a straight line segment $\tau_2 \subseteq B^+_4(Q_2)(v_k) \subseteq B^+_{M/2}(v_k)$ of length $\mathcal{H}^1(\tau_2) < 2\lambda$, one end of which belongs to a free boundary arc with ends on $\alpha_{L,M}$ and the other contained in a free boundary arc with ends on $\alpha_{R,M}$. Moreover, $d(\tau_1, \tau_2) \approx d(Q_1, Q_2) = \epsilon^{-1} d(P_1, P_2)$ can be taken to be of at least unit size

$$d(\tau_1, \tau_2) \geq 1$$

if ϵ is initially taken small enough. However, we can now appeal to Lemma 7.4 below to rule out the arising picture when λ and δ are small enough. This completes the first step of the proof.

Step 2. We have just established that $F(u)$ is smooth everywhere but possibly one point – without loss of generality, this exceptional point sits at the origin. Then each component of $F(u)\setminus 0$ is a smooth submanifold of $\mathbb{R}^2$, hence diffeomorphic to either the circle $\mathbb{S}^1$ or the real line $\mathbb{R}$. Let us establish that the former possibility does not arise. Assume that there is a connected component of $F(u) \setminus 0$ that is a smooth, simple closed curve α. Choose $\delta > 0$ small enough, such that $\mathcal{N}_{2\delta}(\alpha) \cap (F(u) \setminus \alpha) = \emptyset$ (such a δ exists since α is compact and $F(u) \setminus \alpha$ is closed). But since $F(u_k) \to F(u)$ locally in the Hausdorff distance, for $K := \overline{\mathcal{N}_{3\delta/2}(\alpha)}$

$$F(u_k) \cap K \subseteq \mathcal{N}_\delta(F(u) \cap K) = \mathcal{N}_\delta(\alpha)$$

for all k large enough. However, this is impossible, since by the topological assumption (3), the free boundary of u_k has to "exit" $\mathcal{N}_\delta(\alpha)$:

$$(F(u_k) \cap K) \setminus \mathcal{N}_\delta(\alpha) = F(u_k) \cap \left(\overline{\mathcal{N}_{3\delta/2}(\alpha)} \setminus \mathcal{N}_\delta(\alpha)\right) \neq \emptyset.$$

This places us in the assumptions of Lemma 7.5 below, through which we establish the smoothness of $F(u)$ everywhere.

Step 3. Each connected component of $F(u)$ is diffeomorphic to $\mathbb{R}$ and thus, the image of some embedding $\beta : \mathbb{R} \to \mathbb{R}^2$. The embedding has to be proper, i.e. $\lim_{t\to\pm\infty} \beta(t) = \infty$. Otherwise, there exists a sequence, say $t_i \to \infty$, such that $\lim_{i\to\infty} \beta(t_i) = Q \in \mathbb{R}^2$. But $Q \in F(u)$ as $F(u)$ is closed, and since $F(u)$ is smooth at Q, for a small enough $r > 0$ $F(u) \cap \overline{B_r}(Q)$ is a connected arc $\tilde{\beta}$ that contains Q in its interior. But then it has to be that $\tilde{\beta} \cap \beta \neq \emptyset$, so that by connectedness $\tilde{\beta} \subseteq \beta$. The last statement contradicts the finite limit of $\{\gamma(t_i)\}$. Therefore, each connected component of $F(u)$ is a smooth curve, which is the image of a proper embedding of $\mathbb{R}$ into the plane – we shall call such curves "proper arcs". Furthermore, each proper arc of $F(u)$ has strictly positive curvature.

In this last step of the proof of the Proposition, we shall show that $F(u)$ consists of precisely two proper arcs. Then we can invoke [Tra14, Theorem 12] to conclude that u is the hairpin solution.

As we know, u blows down to a wedge solution $s|x_2|$, so for a sequence of $\delta_j \searrow 0$ we can find a sequence $R_j \nearrow \infty$, so that $F(u) \cap B_{R_j} \subseteq \{|x_2| < \delta_j R_j\}$ and $\{x_2 = 0\} \cap B_{R_j} \subseteq \mathcal{N}_{\delta_j R_j}(F(u))$. Define

$$\alpha_{L,j} = \partial B_{R_j} \cap \{x_1 < 0, |x_2| < \delta_j R_j\} \qquad \alpha_{R,j} = \partial B_{R_j} \cap \{x_1 > 0, |x_2| < \delta_j R_j\}.$$

Claim that each connected arc $\gamma \in F(u)$ that intersects B_{R_j} "enters and exits" B_{R_j} either through $\alpha_{L,j}$ or $\alpha_{R,j}$ if R_j is large enough, i.e.

$$\partial B_{R_j} \cap \gamma \subseteq \alpha_{L,j} \quad \text{or} \quad \partial B_{R_j} \cap \gamma \subseteq \alpha_{R,j}.$$

If not, then let U be the connected component of $\{u > 0\}$ such that $\gamma \subseteq \partial U$. Then it's easy to see that $u1_U$ is a viscosity solution of (1) whose free boundary is asymptotically flat, and thus $u1_U$ has to be a half-plane solution, which is impossible since $|\nabla u| < 1$.

As a consequence of the above argument, combined with the fact that $F(u_{R_j}) \to \{x_2 = 0\}$ locally in the Hausdorff distance, it must be that $F(u)$ consists of at least two proper arcs. Assume that $F(u)$ has at least three: γ_1, γ_2 and γ_3, and let j_0 be large enough such that $\partial B_{R_{j_0}} \cap \gamma_i \subseteq \alpha_{L,j_0}$ or $\partial B_{R_{j_0}} \cap \gamma_i \subseteq \alpha_{R,j_0}$, $i = 1, 2, 3$. Note that because γ_i has positive curvature, if say $\partial B_{R_{j_0}} \cap \gamma_i \subseteq \alpha_{L,j_0}$, then $\partial B_{R_j} \cap \gamma_i \subseteq \alpha_{L,j}$ for all $j \geq j_0$ (similarly, if γ_i "enters and exits from the right" at scale R_{j_0}, it will do so at any larger scale). It has to be that at least two of these arcs "enter and exit" from the same side, say γ_1 and γ_2 "enter and exit" from the right. Let $0 < M < R_j/3$ be large enough such that $\{x_1 = M\}$ intersects both γ_1 and γ_2, and consider any connected component V of

$$\{u > 0\} \cap \{M < x_1 < M + R_j/3\}.$$

Applying Lemma 6.5 to the piecewise C^1 Jordan domain V,

$$\mathcal{H}^1(\partial V \cap F(u)) \leq C\mathcal{H}^1(\partial V \setminus F(u)) \leq 4\delta_j R_j,$$

while clearly $\mathcal{H}^1(\partial V \cap F(u)) \geq 2R_j/3$. This leads to a contradiction when $j \to \infty$ as $\delta_j \to 0$.

□

The proof of the proposition is complete modulo the following three technical lemmas.

Lemma 7.4. *Let u be a classical solution of* (1) *in B_{2M} for some large M, whose Lipschitz norm is L. Assume that* (3) *is satisfied and that*

$$\{u = 0\} \cap \overline{B_M} \subseteq \{|x_2| < \delta M\}$$

for some $\delta > 0$. Assume further that $F(u) \cap \overline{B_M}$ consists of arcs, each having its two ends either in α_L or in α_R, where

$$\alpha_L = \partial B_M \cap \{x_1 < 0, |x_2| < \delta M\} \qquad \alpha_R = \partial B_M \cap \{x_1 > 0, |x_2| < \delta M\}.$$

Let F_L (resp. F_R) denote the set of points of $F(u) \cap \overline{B_M}$ that lie on arcs both whose ends belong to α_L (resp. α_R). Then there exist small positive $\delta_0 = \delta_0(L)$ and $\lambda = \lambda(L) < 1$ such that if $0 < \delta < \delta_0$, one cannot find two straight-line open segments τ_1 and τ_2 of length less that λ in $B^+_{M/2}(u)$, each having one end in F_L and one in F_R, and such that dist$(\tau_1, \tau_2) \geq 1$.

Proof. Assume that for some δ, λ such segments exist; we'll derive a contradiction by taking δ and λ small enough and universal. Let τ_1 have ends $p_L \in F_L$ and $p_R \in F_R$, and τ_2 connect $q_L \in F_L$ to $q_R \in F_R$. The following three different scenarios regarding the relation between these points may hold.

Scenario 1. The points p_L, p_R, q_L, q_R belong to distinct arcs in F_L and F_R: $\gamma_{p,L}$, $\gamma_{p,R}$, $\gamma_{q,L}$ and $\gamma_{q,R}$, respectively. Each of these arcs is divided into two subarcs by its respective point – that start on the point and end on ∂B_M; let us choose one of these two and denote it by $\gamma'_{[\cdot],[\cdot]}$, say our choice of a subarc on $\gamma_{p,L}$ will be denoted by $\gamma'_{p,L}$. Then note that $\Gamma_p := \gamma'_{p,L} \cup \tau_1 \cup \gamma'_{p,R}$ and $\Gamma_q := \gamma'_{q,L} \cup \tau_2 \cup \gamma'_{q,R}$ are disjoint simple curves and $B_M \setminus (\Gamma_p \cup \Gamma_q)$ consists of three connected components, only one of which is contained in $\{|x_2| < \delta M\} \cap B_M$; let us call it D. Consider a connected component U of $D^+(u)$ that has τ_1 as part of its boundary. Obviously, U is piecewise C^1 with

$$\mathcal{H}^1(F(u) \cap \partial U) \geq 2M\sqrt{1-\delta^2} - \mathcal{H}^1(\tau_1) \geq 2M\sqrt{1-\delta^2} - \lambda \quad \text{and} \quad \mathcal{H}^1(\partial U \setminus F(u)) \leq 4\arcsin(\delta)M$$

Applying Lemma 6.5 to U, we reach the inequality

$$2M\sqrt{1-\delta^2} - \lambda \leq 4\arcsin(\delta)ML,$$

which cannot be satisfied if δ and λ are small enough.

Scenario 2. Two of the points that "connect to one side" belong to the same arc, while their counterparts belong to distinct arcs: say the points p_L and q_L belong to the same arc γ_L in F_L, while p_R and q_R belongs to two distinct arcs $\gamma_{p,R}$ and $\gamma_{q,R}$, respectively, in F_R. This time let γ'_L denote the subarc of γ_L whose ends are p_L and q_L and let $\gamma'_{p,R}$, $\gamma'_{q,R}$ be determined in the same fashion as in Scenario 1 above. Then $\Gamma := \gamma'_{p,R} \cup \tau_1 \cup \gamma'_L \cup \tau_2 \cup \gamma'_{q,R}$ is a simple curve in B_M with ends on α_R, so that $B_M \setminus \Gamma$ consists of two connected components, only one of which is contained in $\{|x_2| < \delta M\}$; let us again call it D. Consider a connected component U of $D^+(u)$ that has τ_1 as part of its boundary (and thus τ_2 and γ'_L). Then U is piecewise C^1 with

$$\mathcal{H}^1(F(u) \cap \partial U) \geq \mathcal{H}^1(\gamma'_L) + \min\{\mathcal{H}^1(\gamma'_{p,R}), \mathcal{H}^1(\gamma_{p,R} \setminus \gamma'_{p,R})\} + \min\{\mathcal{H}^1(\gamma'_{q,R}), \mathcal{H}^1(\gamma_{q,R} \setminus \gamma'_{q,R})\}$$

since it is either $\gamma'_{p,R}$ or $(\gamma_{p,R} \setminus \gamma'_{p,R})$ (and similarly, $\gamma'_{q,R}$ or $(\gamma_{q,R} \setminus \gamma'_{q,R})$) that belongs to ∂U. But each of these curves intersects both $\partial B_{M/2}$ and ∂B_M, so that its length is at least $M/2$. As $\mathcal{H}^1(\gamma'_L) \geq \operatorname{dist}(\tau_1, \tau_2) \geq 1$, we get

$$\mathcal{H}^1(F(u) \cap \partial U) \geq 1 + M,$$

while on the other hand

$$\mathcal{H}^1(\partial U \setminus F(u)) \leq 2\arcsin(\delta)M.$$

Applying Lemma 6.5 to U we see that

$$1 + M \leq 2\arcsin(\delta)ML,$$

which is violated when δ is small enough.

Scenario 3. In this last scenario, p_L and q_L belong to the same arc γ_L of F_L, and p_R and q_R belong to the same arc γ_R of F_R. Let γ'_L denote the subarc of γ_L with ends p_L, q_L and γ'_R denote the subarc of γ_R with ends p_R, q_R. Then $\Gamma := \tau_1 \cup \gamma'_L \cup \tau_2 \cup \gamma'_R$ is a simple closed curve that encloses a piecewise C^1 Jordan domain $U \subseteq B^+_M(u)$ with

$$\mathcal{H}^1(F(u) \cap \partial U) = \mathcal{H}^1(\gamma'_L) + \mathcal{H}^1(\gamma'_R) \geq 2\operatorname{dist}(\tau_1, \tau_2) \geq 2,$$

while

$$\mathcal{H}^1(\partial U \setminus F(u)) = \mathcal{H}^1(\tau_1) + \mathcal{H}^1(\tau_2) \leq 2\lambda.$$

Lemma 6.5 then yields

$$2 \leq 2\lambda L,$$

which is impossible if λ is small enough. $\square$

Lemma 7.5. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a non-degenerate viscosity and variational solution of* (1) *with $|\nabla u| < 1$ in $\{u > 0\}$. Assume further that $F(u)$ is smooth everywhere but possibly the origin $0 \in F(u)$ and that $F(u) \setminus 0$ has no connected component that is a closed curve. Then $F(u)$ is smooth at the origin as well, and u is a classical solution of* (1) *globally.*

Proof. Every connected component γ of $F(u) \setminus 0$ is a smooth connected submanifold of $\mathbb{R}^2$, and since it is not diffeomorphic to circle $\mathbb{S}^1$ by hypothesis, it has to be diffeomorphic to the real line $\mathbb{R}$. Thus, it is the image of an embedding $\gamma : \mathbb{R} \to \mathbb{R}^2$ and it must be that for any sequence $t_n \to \infty$ (or $t_n \to -\infty$) $\lim_{n\to\infty} \gamma(t_n)$ is either ∞ or 0. Otherwise, there would be some sequence $t_n \to \pm\infty$ such that $\gamma(t_n)$ converges to some finite limit $q \in F(u)$, where $q \neq 0$, because $F(u)$ is closed. But $F(u)$ is smooth at q, so that for a small enough

$r > 0$, $F(u) \cap \overline{B_r}(q)$ is a connected arc β that contains q in its interior. Since $\beta \cap \gamma \neq \emptyset$, it follows that $\beta \subseteq \gamma$ by connectedness, which contradicts the convergence $\gamma(t_n) \to q$.

As a first step, we claim that $F(u) \cap \partial B_1$ consists of finitely many points. Otherwise, there would exist a sequence of points $p_n \in F(u) \cap \partial B_1$ that converges to some $p \in F(u) \cap \partial B_1$; denote by γ the connected component of $F(u) \setminus 0$ containing p. Since $F(u)$ is smooth at p it would have to be that for all large enough n, $p_n \in \gamma$. However, that contradicts the fact that γ is an analytic curve different from the circle ∂B_1.

Second, assume that $F(u) \setminus 0$ has a connected component α, an image of a smooth $\alpha : \mathbb{R} \to \mathbb{R}^2$, such that $\lim_{t \to \pm\infty} \alpha(t) = 0$. Then $\overline{\alpha}$ is a simple closed curve, so that it encloses a bounded connected domain U. Obviously $U \subseteq \{u = 0\}$, as the Strong Maximimum Principle prevents U from containing points x where $u(x) > 0$. Because of Corollary 6.4, α has positive curvature, so we can apply Lemma 7.6 to conclude that $\{u = 0\} \supseteq U$ contains a non-trivial sector based at 0. As a result, the blow-up of u at 0 cannot be the wedge solution and can only be the one-plane solution. Therefore $F(u)$ has to be smooth at 0.

Thus, we may assume that for each connected component γ of $F(u) \setminus 0$, $\gamma(t_n) \to \infty$ for some sequence $t_n \to \infty$ or $t_n \to -\infty$. In particular, each connected component that intersects the unit ball B_1 will exit it at least once. Thus, there are finitely many such connected components, as $F(u) \cap \partial B_1$ consists of finitely many points. Furthermore, the very same reason implies that it is impossible to have $\gamma(t_n) \to \infty$ for one sequence $t_n \to \infty$ (or $-\infty$), while $\gamma(\tilde{t}_n) \to 0$ for another sequence $\tilde{t}_n \to \infty$ (or $-\infty$). Thus, either $\lim_{t \to \pm\infty} \gamma(t) = 0$ or $\lim_{t \to \pm\infty} \gamma(t) = \infty$.

Next we note that 0 cannot be an isolated point of $F(u)$, so there exists a sequence of points $P_n \in F(u)$ such that $P_n \to 0$. Since there are only finitely many connected components of $F(u) \setminus 0$ intersecting B_1, there exists a subsequence P_{n_k} that belongs to a single connected component γ_1, so that $\lim_{t \to \infty} \gamma_1(t) = 0$. Then it must be $\lim_{t \to -\infty} \gamma_1(t) = \infty$, so the latest "entry time" for γ_1 into B_1,

$$T := \sup\{t : \gamma_1(t) \in (B_1)^c\}$$

satisfies $|T| < \infty$. Consider the connected component V of $(\{u = 0\} \cap B_1)^\circ$ having $\gamma_1\big([T, \infty]\big)$ as part of it boundary. Obviously, $0 \in \partial V$ and claim that there exists another curve $\gamma_2 : (-\infty, 0] \to \mathbb{R}^2$ such that $\gamma_2\big((-\infty, 0]\big) \subset \partial V \cap (F(u) \setminus 0)$ with $\lim_{t \to -\infty} \gamma_2(t) = 0$. If not, $\Big(\partial V \setminus \gamma_1\big([T, \infty]\big)\Big) \cap F(u)$ consists of finitely many free boundary arcs with ends p_{2k} and p_{2k+1} on the unit circle ∂B_1, $k = 0, 1, 2, \ldots, l$. Here we have chosen the enumeration of the points $\{p_i\}$ in such a way that the shorter circular arc $\widehat{p_1 p_2} \subseteq \partial V$, and that has p_{i+1} following p_i in the direction (clockwise or counterclockwise) set by p_1 and p_2. In this way, the circular arcs $\widehat{p_{2k+1} p_{2k+2}} \subseteq \partial V$, $k = 0, 1, \ldots, l-1$. Let $q \in F(u) \cap \partial B_1$ be the next point after p_{2l+1} on the unit circle as we traverse it in the same direction. Then it must be that $\widehat{p_{2l+1} q} \subseteq \partial V$ and that the connected component of $(F(u) \setminus 0) \cap \overline{B_1}$, having q as one of its ends, is also part of ∂V. Since the other end of that component can neither lie on ∂B_1 nor be 0, it has to be that $q = p_0$. This is, however, impossible as u cannot be zero on both sides of γ_1. So, there exists a free boundary curve $\gamma_2 \subset \partial V \cap (F(u) \setminus 0)$, disjoint from γ_1, with $\gamma_2(-\infty) = 0 = \gamma_1(\infty)$. From here it is not hard to see that V is a Jordan domain. Again, Corollary 6.4 says that γ_1 and γ_2 have positive curvature, and we can invoke Lemma 7.6 to establish that V contains a non-trivial sector based at 0. As before, the blow-up limit of u at zero has to be the half-plane solution, so that $F(u)$ is smooth. □

Lemma 7.6. *Let $U \subseteq \mathbb{R}^2$ be a Jordan domain with $0 \in \partial U$ and let $\gamma_1 \in C^2([0, \infty), \mathbb{R}^2)$ and $\gamma_2 \in C^2((-\infty, 0], \mathbb{R}^2)$ be some regular parameterizations ($\gamma_i' \neq 0$, $i = 1, 2$) of two simple disjoint subarcs of ∂U, for which $\lim_{t \to \infty} \gamma_1(t) = \lim_{t \to -\infty} \gamma_2(t) = 0$, and such that traversing ∂U in the counterclockwise direction corresponds to t increasing. Assume further that their curvatures are strictly positive. Then $B_r \cap U$ contains a non-trivial sector of B_r.*

Proof. First let us introduce some notation. For a point p in γ_i, $i = 1, 2$, let $L(p)$ be the tangent line to γ_i at p. If $p = \gamma_i(t_0)$ for some t_0, let $\tau(p) = \gamma_i'(t_0)/|\gamma_i'(t_0)|$ be the unit tangent vector to p in the direction of $\gamma_i'(t_0)$; let $\nu(p)$ be the unit normal vector to γ_i at p that one gets by rotating $\tau(p)$ by $\pi/2$. Denote by $H^+(p)$ and $H^-(p)$ the two half-planes:

$$H^\pm(p) = \{x :\in \mathbb{R}^2 : (x - p) \cdot (\pm\nu(p)) > 0\}.$$

For any two points $p = \gamma_i(t_1)$ $q = \gamma_i(t_2)$, $t_1 < t_2$, define $\theta(p, q)$ to be the angle $\gamma_i'(t)$ sweeps as t increases from t_1 to t_2. Then the fact that γ_i has positive curvature is equivalent to $\theta(\gamma_i(t), \gamma_i(t+s))$ being a positive,

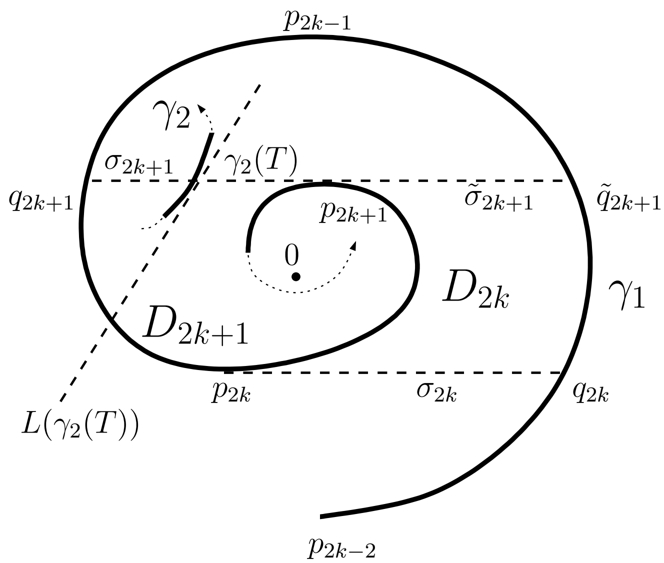

FIGURE 1. The curves γ_1 and γ_2.

strictly increasing function of s for $s > 0$ and any fixed t. For any $p \in \gamma_i(t)$ and $\alpha > 0$, denote by $T^\alpha(p)$ the point $q \in \gamma_i$ such that $\theta(p, q) = \alpha$, if it exists. Let $s(p, q)$ be the open segment of γ_i with ends $p, q \in \gamma_i$.

Now we proceed with the argument. We shall show that $\theta(\gamma_1(0), \gamma_1(t))$ is bounded from above as a function of t. Assume not; then $T^\alpha(p)$ exists for any $p \in \gamma_1$ and any $\alpha > 0$. Claim that there exists a $p_0 \in \gamma_1$ such that $T^{2\pi}(p_0) \in \overline{H^+(p_0)}$. If not, then for any $p \in \gamma_1$ and $k \in \mathbb{N}$,

$$T^{(2k+2)\pi}(p) \in H^-(T^{2k\pi}(p)) \Subset H^-(p) \quad \text{as well as}$$
$$T^{(2k+3)\pi}(p) \in H^-(T^{(2k+1)\pi}(p)) \Subset H^-(T^\pi(p)).$$

However, note that because γ_1 has positive curvature, we have $s(p, T^\pi(p)) \subseteq H^+(p)$, as the smallest $\alpha > 0$ for which $s(p, T^\alpha(p))$ can intersect $L(p)$ must be greater than π. Thus, $H^-(T^\pi(p)) \Subset H^+(p)$. But since $\gamma_1(t) \to 0$ as $t \to \infty$ and $\gamma_1' \neq 0$, then both

$$T^{(2k+2)\pi}(p) \to 0 \quad \text{and} \quad T^{(2k+3)\pi}(p) \to 0 \quad \text{as } k \to \infty.$$

That contradicts the fact that $T^{(2k+2)\pi}(p) \in H^-(p)$ whereas $T^{(2k+3)\pi}(p) \in H^+(p)$.

Thus, for some p_0, $T^{2\pi}(p_0) \in \overline{H^+(p_0)}$, so that the whole segment $s(p_0, T^{2\pi}(p_0)) \subseteq H^+(p_0)$. Denote

$$p_j := T^{j\pi}(p_0) \quad j \in \mathbb{N}.$$

Claim we then have $s(p_{2k-2}, p_{2k}) \subseteq H^+(p_{2k-2})$ for all $k \in \mathbb{N}$. Argue by induction. Note that since $s(p_{2k-1}, p_{2k}) \subseteq H^+(p_{2k-1}) \cap H^+(p_{2k})$, there are exactly two intersection points between $\overline{s}(p_{2k-2}, p_{2k})$ and $L(p_{2k})$, namely p_{2k} and a point $q_{2k} \in \overline{s}(p_{2k-2}, p_{2k-1})$ (see Figure 1). If it were the case that $p_{2k+2} \in H^-(p_{2k})$, the segment $s(p_{2k}, p_{2k+2})$ would have to leave the convex domain $D_{2k} \subseteq H^+(p_{2k}) \cap H^+(p_{2k-1})$, enclosed by $s(q_{2k}, p_{2k})$ and the straight-line segment $\sigma_{2k} := p_{2k}q_{2k}$. But obviously $s(p_{2k}, p_{2k+1}) \subseteq D_{2k}$, so it would have to be $s(p_{2k+1}, p_{2k+2})$ that exits D_{2k}. Construct as above the point $q_{2k+1} \in s(p_{2k-1}, p_{2k})$ being the second intersection point of $L(p_{2k+1})$ with $\overline{s}(p_{2k-1}, p_{2k+1})$ and let $\sigma_{2k+1} \subset D_{2k}$ be the straight-line segment $p_{2k+1}q_{2k+1}$. Then the convex domain D_{2k+1}, enclosed by σ_{2k+1} and $s(q_{2k+1}, p_{2k+1})$, is contained within the convex D_{2k}, so that $s(p_{2k+1}, p_{2k+2})$ which enters D_{2k+1} would have to exit D_{2k+1} before it exits D_{2k}. That is, however, impossible as $s(p_{2k+1}, p_{2k+2}) \subseteq D_{2k+1}$. The induction step is complete.

Let now $K \in \mathbb{N}$ be large enough such that for all $k \geq K$, $s(p_{2k-2}, p_{2k}) \subseteq B_\delta(0)$ where $\delta > 0$ is small enough, such that $\gamma_2(0) \in B_{2\delta}(0)^c$ (such a K exists since $\gamma_1(t) \to 0$ as $t \to \infty$). Since $\gamma_2(t) \in D_{2k+1}$ for all large $|t|$, its last 'time of exit' from D_{2k+1}

$$T := \sup\{t : \gamma_2(t) \in D_{2k+1}\}$$

exists and must satisfy $T < 0$. Obviously, $\gamma_2(T)$ must belong to σ_{2k+1} and γ_2 must intersect σ_{2k+1} transversally, for otherwise the fact that γ_2 has positive curvature would imply that for some small $\epsilon < 0$ $\gamma([T-\epsilon, T+\epsilon])$ would lie on one side of σ_{2k+1} which contradicts the definition of T. Let $\tilde{q}_{2k+1}$ denote the intersection point of $L(p_{2k+1})$ and $s(p_{2k-2}, p_{2k-1})$, and let $\tilde{\sigma}_{2k+1}$ be the straight-line segment $p_{2k+1}\tilde{q}_{2k+1}$. Note that since γ_2 intersects σ_{2k+1} transversally, $\tilde{\sigma}_{2k+1} \subseteq H^-(\gamma_2(T))$. Also, since $\gamma_2(0) \notin B_\delta(0)$, $\gamma_2([T,0])$ must exit the domain $\tilde{D}_{2k+1} \subseteq B_\delta(0)$, enclosed by $s(\tilde{q}_{2k+1}, q_{2k+1})$ and the straightline segment $q_{2k+1}\tilde{q}_{2k+1}$, having once entered it. Thus $\gamma_2([T,0])$ intersects $\tilde{\sigma}_{2k+1} \subseteq H^-(\gamma_2(T)))$, so that $\gamma_2((T,0])$ must cross $L(\gamma_2(T))$. Let T_1 be the first time $\gamma((T,0])$ crosses $L(\gamma_2(T))$:

$$T_1 := \inf\{t > T, \gamma_2(t) \in L(\gamma_2(T))\}.$$

Note that $T_1 > T$ as $\gamma_2((T, T+\delta)) \subseteq H^+(\gamma_2(T)))$ for all small enough $\delta > 0$. Furthermore, it must be that $\theta(\gamma_2(T), \gamma_2(T_1)) \geq \pi$ but $\theta(\gamma_2(T), \gamma_2(T_1)) \leq 2\pi$. The former bound is obvious; the latter is true for otherwise $T^{2\pi}(\gamma_2(T)) \in H^+(\gamma_2(T))$, so that by the same argument as before we would have all of $\gamma_2((T,0]) \subseteq H^+(\gamma_2(T))$, which would prevent it from crossing $L(\gamma_2(T))$. As a result, it must be that

$$\big(\gamma_2(T_1) - \gamma_2(T)\big) \cdot \gamma_2'(T) < 0,$$

which in turn implies that $\gamma_2((T, T_1))$ must cross the straighline segment $\gamma_2(T)q_{2k+1}$, which is impossible.

Therefore, $\theta(\gamma_1(0), \gamma(t))$ is bounded from above, so that $\tau_1 := \lim_{t\to\infty} \gamma_1'(t)/|\gamma_1'(t)|$ exists. Exchanging the roles of γ_1 and γ_2 in the argument above, we can show that $\tau_2 := \lim_{t\to-\infty} \gamma_2'(t)/|\gamma_2'(t)|$ exists, as well. As a result, for all small enough $r > 0$, $\gamma_1 \cap B_r$ and $\gamma_2 \cap B_r$ are flat graphs over the radii along τ_1 and τ_2, respectively. Let $A_i = \partial B_r \cap \gamma_i$, $i = 1, 2$ be the points of intersection of ∂B_r with γ_1 and γ_2. Because of the positivity of the curvature, the open straight-line segments connecting 0 to A_1 and 0 to A_2 are contained in U for r small enough. Also, since $\partial B_r \cap \partial U = \{A_1, A_2\}$, the whole open circular arc $\widehat{A_2A_1}$ (as we trace the circle in the counter-clockwise direction) must be contained in the Jordan domain U. Thus, U contains the entire open circular sector with vertex 0 and arc $\widehat{A_2A_1}$.

□

8. Local structure.

In this section we shall study the shape of the free boundary of solutions of (1), defined in the unit disk, satisfying the topological assumption (3). This will be carried out by examining blow-up limits of sequences of solutions in B_1, for which exact purpose the classification Theorem 1.1 was developed. We encounter the following dichotomy: if a component of the zero phase is well separated by the rest of the zero phase, its boundary has bounded curvature (in terms of the separation) – this is the content of Proposition 8.2 below. Once the separation becomes small enough relative a certain universal scale, we shall see the signs of a hairpin-like structure arising – this is described in Propositions 8.3 and 8.4.

Let us make the following definition for ease of reference.

Definition 8.1. *We shall call the free boundary $F(u)$ of a solution u of* (1) *δ-flat in $B = B_r(p)$ if for some rotation ρ,*

$$|u(p+\rho x) - x_2^+| \leq \delta r \quad \text{for} \quad x \in B_r(0).$$

Remark 8.1. *Denote by δ_0 the small universal constant, such that if $0 < \delta < \delta_0$ small enough, the Alt-Caffarelli regularity theory* [AC81] *states that $F(u) \cap B_{r/2}$ is a graph in the direction of $\rho(e_2)$ with Lipschitz norm at most $C\delta$. For such δ we also have the bound*

$$\|\rho\nabla u - e_2\|_{L^\infty(B^+_{r/2}(u))} + r\|D^2u\|_{L^\infty(B^+_{r/2}(u))} \leq c\delta.$$

It implies, in particular, that the curvature of $F(u)$ in $B_{r_0/2}$ is $O(\delta)$.

The next proposition treats the scenario where a point of the free boundary $F(u)$ is distance at least s away from all other components of the zero phase; then we expect a curvature bound on $F(u)$ at the point.

Proposition 8.2. *Let u be a classical solution of* (1) *in B_1 that satisfies* (3) *and assume $0 \in F(u)$. Denote by Z the connected component of 0 in $\{u = 0\}$. For any $0 < s < 1$ there exists $\kappa = \kappa(s) < \infty$ such that if*

$$d(0, \{u = 0\} \setminus Z) \geq s$$

then the curvature of $F(u)$ at 0 is at most κ.

Proof. Assume the proposition is false. Then we have a sequence of counterexamples u_l for which the curvature κ_l of $F(u_l)$ at zero is

$$\kappa_l \geq l^2.$$

Define the rescales

$$\tilde{u}_l(x) := l u_l(x/l) \quad \text{for} \quad x \in B_l.$$

Then the curvature $\tilde{k}_l$ of $F(\tilde{u}_l)$ at 0 satisfies

$$\tilde{\kappa}_l = \kappa_l / l \geq l. \tag{16}$$

By our classification Theorem 1.1 we see that, up to taking a subsequence, the $\tilde{u}_l$ converge uniformly on compact subsets to a global solution $\tilde{u}$ that is either a one-plane, a two-plane, a hairpin or a wedge solution.

Let $\delta_0 > 0$ be the small universal constant defined in Remark 8.1. If $\tilde{u}$ is a one-plane solution, then for all large enough l, in some Euclidean coordinates

$$|\tilde{u}_l - x_2^+| < \delta_0/2 \quad \text{in} \quad B_1,$$

hence $F(\tilde{u}_l \cap B_1)$ is δ_0-flat and $\tilde{k}_l \leq C\delta_0$ which contradicts (16). Similarly, if $\tilde{u}$ is a two-plane solution, for some $b < 0$ and all large enough l

$$|\tilde{u}_l - (x_2^+ + (x_2 - b)^-)| < \min\{\delta_0/2, b/10\} \quad \text{in} \quad B_1.$$

Thus,

$$v_l := \tilde{u}_l 1_{B_1 \cap \{x_2 > b/2\}}$$

is a classical solution of (1) in B_1, whose free boundary is δ_0-flat in B_1, so $\tilde{\kappa}_l \leq C\delta_0$ – a contradiction.

Analogously, we can rule out $\tilde{u}$ being a hairpin solution. Assume that it is; then we can find a scale s_0 such that for every $p \in F(\tilde{u})$

$$d_H(F(\tilde{u}) \cap B_{s_0}(p), L(x) \cap B_{s_0}(p)) < \delta_0 s_0/2,$$

where $L(p)$ denotes the tangent line to $F(\tilde{u})$ at p. Now for all large enough l,

$$d_H(F(\tilde{u}_l) \cap B_{s_0}, L(0) \cap B_{s_0}) < \delta_0 s_0$$

so that $w_l(y) := \tilde{u}_l(s_0 y)/s_0$ has a δ_0-flat free boundary in B_1 and the curvature of $F(w_l)$ at 0 is bounded by $C\delta_0$. Thus, the curvature of $F(\tilde{u}_l)$ at 0

$$\kappa_l \leq C\delta_0/s_0,$$

which again contradicts (16).

Finally, assume that $\tilde{u} = |x_2|$ is the wedge-solution. Then for all l large enough

$$d_H(F(\tilde{u}_l) \cap \overline{B_4}, \{x_2 = 0\} \cap \overline{B_4}) \leq \delta_0. \tag{17}$$

Let $N = (0, 1)$ and $S = (0, -1)$. Note that N and S cannot belong to two separate components of $B_2^+(\tilde{u}_l)$, for according to Lemma 6.7, $F(\tilde{u}) \cap \{|x_1| < 1/2\} \cap B_4$ consists of two graphs of Lipschitz norm at most $c\delta_0$, so that we again get an upper bound for $\tilde{k}_l$ for all large l. This means that if $F(\tilde{u}_l) \cap \overline{B_3}$ consists of finitely many arcs, each of which "attaches" either to α_L or α_R, where

$$\alpha_L = \partial B_3 \cap \{x_1 < 0, |x_2| < \delta_0\} \qquad \alpha_R = \partial B_3 \cap \{x_1 > 0, |x_2| < \delta_0\}.$$

Thus, if F_L (F_R) denotes the union of the arcs of $F(\tilde{u}_l) \cap \overline{B_3}$ that attach to α_L (α_R), then F_L and F_R are disjoint compact sets and so $d(F_L, F_R)$ is realized for some $p \in F_L$ and $q \in F_R$. Moreover, the straight line (open) segment τ with ends p and q is contained in $B_3^+(\tilde{u}_l)$ and because of (17), we have

$$|p - q| = \mathcal{H}^1(\tau) \leq 6\delta_0.$$

On the other hand, note that if $\tilde{Z}_l$ denotes the connected component of 0 in $\{\tilde{u}_l = 0\}$ in B_l, we have by assumption

$$d(0, \{\tilde{u}_l = 0\} \setminus \tilde{Z}_l) \geq ls \gg 1,$$

hence it must be that both p and q belong to the same boundary arc of $\partial\tilde{Z}_l$ (they cannot belong to different boundary arcs of $\partial\tilde{Z}_l$, for p and q would have to lie on the boundary of two different connected components of $B_l^+(\tilde{u})$). Let $\beta \subseteq F(\tilde{u}_l)$ denote the arc connecting p to q. Then $\beta \cap \tau$ encloses a piecewise-C^2 Jordan domain $V \subseteq B_l^+(\tilde{u}_l)$ and applying Lemma 6.5 to V, we find that

$$6\delta_0 \geq \mathcal{H}^1(\tau) \geq c\mathcal{H}^1(\beta)$$

which is impossible for small δ_0 as $\mathcal{H}^1(\beta) \geq 2$. This completes the proof. $\square$

Proposition 8.3. *Let u be a classical solution of* (1) *in B_1 that satisfies* (3) *and assume $0 \in F(u)$. Denote by Z the connected component of 0 in $\{u = 0\}$. Then for any $0 < \delta < \delta_0$ small enough there exist $0 < \epsilon_0 \ll 1$ and $r_0 > 0$ such that if for any one $0 < r \leq r_0$*

$$d(0, \{u = 0\} \setminus Z) < \epsilon_0 r$$

then for some rotation ρ,

$$|u(\rho x) - |x_2|| < \delta r \quad \text{in } B_r.$$

Proof. Fix $0 < \delta < \delta_0$. By the scale-invariance of the problem it suffices to show the conclusion of the proposition holds only for $r = r_0$. Assume not; then for any sequences of $\epsilon_k \to 0$, $r_k \to 0$, there exists a corresponding sequence of counterexamples u_k in B_1: namely, if Z_k denotes the component of 0 in $\{u_k = 0\}$, we have $d(0, \{u_k = 0\} \setminus Z_k) \leq \epsilon_k r_k$, but

$$\|u_k(\rho x) - |x_2|\|_{L^\infty(B_{r_k})} > \delta r_k \tag{18}$$

for all rotations ρ. Define the rescaled

$$\tilde{u}_k(x) := u_k(r_k x)/r_k \quad \text{in } B_{1/r_k}.$$

According to the Classification Theorem 1.1, up to taking a subsequence, $\tilde{u}_k$ converge uniformly on compact subsets of $\mathbb{R}^2$ to $\tilde{u}$, being either the half-plane, the wedge, a two-plane or a hairpin solution.

If $\tilde{u} = x_2^+$ in an appropriate coordinate system, then for all large enough k

$$\{\tilde{u}_k = 0\} \cap B_1 \cap \{|x_1| < 1/2\} = \{x \in B_1 : |x_1| < 1/2, x_2 < \phi(x_1)\}$$

for some $C\delta_0$–Lipschitz function $\phi : (-1/2, 1/2) \to \mathbb{R}$. In particular $\{\tilde{u}_k = 0\} \cap B_{1/2}$ consists of a single component (the one containing 0). Hence, going back to the original scale,

$$d(0, \{u_k = 0\} \setminus Z_k) \geq r_k/2 > \epsilon_k r_k$$

for k large enough, which contradicts our assumption. Similarly, we rule out the case when $\tilde{u}$ is the two-plane solution. If $\tilde{u}$ is a hairpin, we can find a scale s_0, such that for every $x \in F(\tilde{u})$

$$d_H(F(\tilde{u}) \cap B_{s_0}(x), L(x) \cap B_{s_0}(x)) < \delta_0 s_0/2,$$

where $L(x)$ denotes the tangent line through x to the hairpin $F(\tilde{u})$. Then, for all large enough k,

$$d_H(F(\tilde{u}_k) \cap B_{s_0}, L(0) \cap B_{s_0}) \leq d_H(F(\tilde{u}) \cap B_{s_0}, L(0) \cap B_{s_0}) + d_H(F(\tilde{u}_k) \cap B_{s_0}, F(\tilde{u}) \cap B_{s_0}) \leq s_0\delta_0/2 + s_0\delta_0/2 = s_0\delta_0,$$

so that in $B_{s_0/2}$, $\{\tilde{u}_k = 0\} \cap B_{s_0/2}$ consists of a single component. Going back to scale r_k, we see that

$$d(0, \{u_k = 0\} \setminus Z_k) \geq s_0 r_k/2 > \epsilon_k r_k$$

which is a contradiction when k is large enough.

Therefore, $\tilde{u}$ must be the wedge solution: $\tilde{u} = |x_2|$ in an appropriately rotated coordinate system. This leads to a contradiction with (18), however, because it implies that for all k large enough, we actually have

$$\|u_k(x) - |x_2|\|_{L^\infty(B_{r_k})} \leq \delta r_k.$$

$\square$

Proposition 8.4. *For any given $0 < \delta < \delta_0$, let ϵ_0, r_0 and $u : B_1 \to \mathbb{R}$ be as in Proposition 8.3. Let Z denote the component of 0 in $\{u = 0\}$. Then for any $0 < r \leq r_0$ such that*

$$d(0, \{u = 0\} \setminus Z) < \epsilon_0 r,$$

the free boundary $F(u) \cap \overline{B_{r/2}}$ consists of exactly two arcs $F_L \subseteq Z$ and $F_R \subseteq \{u = 0\} \setminus Z$. Those are contained in $\rho(S_{r/2,\delta r})$ for an appropriate rotation $\rho = \rho_r$, where

$$S_{r,t} := \{x \in \mathbb{R}^2 : |x_1| \leq r, |x_2| \leq t\},$$

with the two ends of F_L in $\rho(\alpha_{L,r/2})$ and the two ends of F_R in $\rho(\alpha_{R,r/2})$, where

$$\alpha_{L,r} = \{x \in \partial B_r : x_1 < 0, |x_2| < \delta r\} \quad \text{and} \quad \alpha_{R,r} = \{x \in \partial B_r : x_1 > 0, |x_2| < \delta r\}.$$

Moreover, the minimum distance between the corresponding two components of $\{u = 0\} \cap \overline{B_{r/2}}$ is realized for some points $p \in F_L$, $q \in F_R$ with both $p, q \in \rho(S_{r/3,\delta r})$.

Proof. Fix r and choose Euclidean coordinates appropriately so that ρ_r is the identity. Let γ be the arc of $F(u) \cap B_r$, containing 0. Claim that the two ends of γ both belong to either $\alpha_{L,r}$ or $\alpha_{R,r}$. Assume not. Then the points $N = (0, r/2)$ and $S = (0, r/2)$ belong to two distinct connected components of $B_r^+(u)$, so that according to Lemma 6.7, $F(u) \cap B_r \cap \{|x_1| < r/4\}$ consists of two disjoint graphs $\Sigma_\pm = \{x_2 = \phi_\pm(x_1) : |x_1| < r/4\}$ of Lipschitz norm at most $C\delta$ and

$$u(x) = 0 \quad \text{for} \quad x \in \{\phi_-(x_1) \leq x_2 \leq \phi_+(x_1) : |x_1| < r/4\}$$

But then $d(0, \{u = 0\} \setminus Z) \geq r/4 > \epsilon_0 r$, which contradicts our hypothesis. Hence, we may assume that γ attaches on $\alpha_{L,r}$.

Look now at the free boundary in $B_{r/3}(P)$, where $P = (-r/2, 0)$. Since $\gamma \subseteq S_{r,\delta r}$ connects α_L to 0, it must be that γ disconnects $\tilde{N}_L := P + (0, r/6)$ from $\tilde{S}_L := P + (0, -r/6)$ in $B_{r/2}(p)^+(u)$. Invoking Lemma 6.7 again, we see that $F(u) \cap B_{r/3}(P) \cap \{|x_1 + r/2| < r/6\}$ consists of two graphs of Lipschitz norm at most $C\delta$. As a result, the connected component $\tilde{Z}_L$ of 0 in $\{u = 0\} \cap B_{r/2}$ is bounded by a single free boundary arc F_L and a circular subarc of $\alpha_{L,r/2}$ that share ends. Another even more significant consequence is that $F(u) \cap B_r$ contains no other arcs besides γ that intersect $V_L := B_r \cap \{|x_1 + r/2| < r/6\}$. Since $\{u = 0\}$ has a component different from Z that is at most $\epsilon_0 r$ away from 0, $F(u) \cap B_r$ contains at least one more arc $\tilde{\gamma} \neq \gamma$. According to the observation above, $\tilde{\gamma}$ doesn't cross into the region V, so it has to attach on $\alpha_{R,r}$. Consider $F(u) \cap B_{r/3}(Q)$, where $Q = (r/2, 0)$. Since $\tilde{\gamma} \cap B_{\epsilon_0 r} \neq \emptyset$, it must be that $\tilde{\gamma}$ disconnects $\tilde{N}_R = Q + (0, r/6)$ from $\tilde{S}_R = Q + (0, -r/6)$ in $B_{r/3}(Q)^+(u)$. Thus, by Lemma 6.7, $F(u) \cap B_{r/3}(Q) \cap \{|x_1 - r/2| < r/6\}$ consists of two graphs of Lipschitz norm at most $C\delta$. Hence, $\{u = 0\} \cap B_{r/2}$ has only one other connected component $\tilde{Z}_R$ and $\partial Z_R \cap F(u)$ consists of a single free boundary arc F_R. As F_R cannot intersect V_L and, similarly, F_L cannot intersect $V_R := B_r \cap \{|x_1 - r/2| < r/6\}$, it must be that the minimum distance between $\tilde{Z}_L$ and $\tilde{Z}_R$ is realized for some points $p \in F_L$ and $q \in F_R$ with $|x_1(p)| < r/2 - r/6 = r/3$ and $|x_1(q)| < r/3$. □

Remark 8.5. *Assume we are in the situation of Propositions 8.3 and 8.4 above for some fixed small* $0 < \delta < \delta_0$. *Then* $F(u) \cap \overline{B_{r_0/2}}$ *consists of two arcs* F_L *and* F_R*, and the minimum distance* $s = d(F_L, F_R)$ *is realized for some points* $p \in F_L$, $q \in F_R$ *with both* $p, q \in \rho_{r_0}(S_{r_0/3, \delta r_0})$. *Now apply again Propositions 8.3 and 8.4 to the translate*

$$\tilde{u}(y) := u(p + y) \quad y \in B_{1/2}.$$

Call $\tilde{Z}$ *the connected component of* $\{\tilde{u} = 0\}$ *containing* 0. *We establish that for every* r *such that* $s/\epsilon_0 < r \leq r_0$*, there is a rotation* $\tilde{\rho} = \tilde{\rho}_r$ *such that*

$$|\tilde{u}(\tilde{\rho} y) - |y_2|| < \delta r \quad \text{in } B_r$$

and the free boundary in $B_{r/2}$, $F(\tilde{u}) \cap \overline{B_{r/2}} \subseteq \tilde{\rho}(S_{r/2, \delta r})$ *consists of two arcs* $\tilde{F}_L \subseteq \tilde{Z}$ *and* $\tilde{F}_R \subseteq \{\tilde{u} = 0\} \setminus Z$, *the minimum distance between which is realized for* $0 \in \tilde{F}_L$ *and* $q - p \in \tilde{F}_R$.

9. Lipschitz bound of free boundary strands.

In this section we shall further elaborate on the finer-scale structure of the free boundary of a solution that falls under the scenario of Proposition 8.3. More specifically, we shall show that if the separation s between two components of the zero phase becomes small enough, it forces the free boundary outside that scale to be the union of four graphs of small Lipschitz constant over a common line.

Theorem 9.1. *For any given small* $0 < \delta < \delta_0$, *there exist* $r_0 > 0$, $\epsilon_0 > 0$ *such that if* u *is a classical solution of* (1) *in* B_1, *satisfying* (3), *with* $0 \in F(u)$ *and*

$$\mathit{dist}(0, \{u = 0\} \setminus Z) < \epsilon_0 r_0,$$

then for some $p \in B_{r_0/3}$, $B_{r_0/2}(p) \cap F(u)$ *consists of two free boundary arcs* F_L *and* F_R*, the shortest segment between which is centered at* p*, the separation*

$$s := \mathit{dist}(F_L, F_R) < \epsilon_0 r_0.$$

and for some rotation ρ *and functions* $f, g : \mathbb{R} \to \mathbb{R}$ *with* $f < g$

$$\{u = 0\} \cap \big(B_{r_0/2}(p) \setminus B_{4s/\epsilon_0}(p)\big) = p + \rho\{4s/\epsilon_0 < |x| < r_0/2 : f(x_1) \leq |x_2| \leq g(x_1)\}$$
$$\text{where} \quad \|f\|_{L^\infty} + \|g\|_{L^\infty} \leq \delta r, \quad \|f'\|_{L^\infty} + \|g'\|_{L^\infty} \leq \delta.$$

That is, $F(u) \cap \big(B_{r_0/2}(p) \setminus B_{4s/\epsilon_0}(p)\big)$ consists of four graphs over a common line with Lipschitz norm at most δ.

The proof will be carried out in Lemmas 9.2 and 9.3 below. Assume δ, r_0, ϵ_0 are as in Proposition 8.3. In view of Remark 8.5, we may assume that we are dealing with a solution of (1) in B_{r_0}, which satisfies:

- $F(u) \cap \overline{B_{r_0/2}}$ consists of two arcs F_L and F_R; for some rotation ρ_{r_0} the ends of F_L belong to $\rho_{r_0}(\alpha_{L,r_0/2})$ and the ends of F_R belong to $\rho_{r_0}(\alpha_{R,r_0/2})$, where
$$\alpha_{L,r} = \{x \in \partial B_r : x_1 < 0, |x_2| < \delta r\} \quad \text{and} \quad \alpha_{R,r} = \{x \in \partial B_r : x_1 > 0, |x_2| < \delta r\}.$$
- The minimum distance $d(F_L, F_R) = s$ is realized for $0 \in F_L$ and some point $q \in F_R$ with $0 < s < \epsilon_0 r_0$.
- For every $s/\epsilon_0 < r \leq r_0$,
$$|u(\rho y) - |y_2|| < \delta r \quad \text{in } B_r \quad \text{for some rotation } \rho = \rho_r.$$
- For every $s/\epsilon_0 < r \leq r_0/2$, $F(u) \cap \overline{B_r}$ consists of two arcs $F_L(r)$ and $F_R(r)$ that attach on $\rho_r(\alpha_{L,r})$ and $\rho_r(\alpha_{R,r})$.

Set
$$r_k := 2^{k-1} s/\epsilon_0 \qquad k \in \mathbb{N}$$
and let $k_0 = \lfloor \log_2(r_0\epsilon_0/s) \rfloor$, so that $r_{k_0} \approx r_0/2$. Define F_L^N and F_L^S to be the two (closed) subarcs of $F_L(r_{k_0})$ that 0 divides $F_L(r_{k_0})$ into: with F_L^N being the one such that the end point $\rho_{r_{k_0}}^{-1}(F_L^N) \cap \alpha_{L,r_{k_0}}$ has the greater x_2-coordinate than the end point $\rho_{r_{k_0}}^{-1}(F_L^S) \cap \alpha_{L,r_{k_0}}$. Define F_R^N and F_R^S, the two subarcs of $F_R(r_{k_0})$ that q divides $F_R(r_{k_0})$ into, analogously. Let τ be the straight-line close segment connecting 0 to q, and let β^N and β^S be the two circular arcs of $\partial B_{r_{k_0}} \cap \{u > 0\}$ with β^N containing $\rho_{r_{k_0}}\big((0, r_{k_0})\big)$ and β^S containing $\rho_{r_{k_0}}\big((0, -r_{k_0})\big)$. Then τ splits $B^+_{r_{k_0}}(u)$ into two simply-connected regions – the "top" Ω_N, bounded by β^N, F_L^N, τ, F_R^N; and the "bottom" Ω_S, bounded by β^S, F_L^S, τ, F_R^S.

We may choose the coordinate system so that ρ_{r_1} is the identity. In the following series of arguments we shall adopt complex notation: denoting the point $(x_1, x_2) \in \mathbb{R}^2$ by the complex $z = x_1 + ix_2 \in \mathbb{C}$.

Let $z_k \in \mathbb{C}$ be the unique point of intersection between ∂B_{r_k} and F_R^N, $k = 1, 2, \ldots, k_0$. The region Ω_N is simply connected with piece-wise smooth boundary, so we may define the harmonic conjugate $v : \Omega_N \to \mathbb{R}$ of u, such that v is continuous up to the boundary $\partial\Omega_N$ and has the normalization
$$v(z_2) = -|z_2|.$$
Now define the holomorphic map $U : \Omega_N \to \mathbb{C}$ by
$$U := iu - v.$$

Lemma 9.2. *The map U constructed above is injective on $\overline{\Omega_N \setminus B_{r_2}}$ and its image*
$$U(\overline{\Omega_N \setminus B_{r_2}}) \supseteq \{\xi \in \mathbb{C} : Im(\xi) \geq 0, r_2(1 + C\delta) \leq |\xi| \leq r_{k_0}(1 - C\delta)\} \tag{19}$$
for some numerical constant C.

Proof. First, let us note that for $k = 2, \ldots, k_0 - 1$, the free boundary in each dyadic annulus $F(u) \cap \overline{B_{r_{k+1}} \setminus B_{r_k}}$ consists of four graphs of Lipschitz norm at most $c'\delta$ for some numerical constant $c' > 0$. This is a direct consequence of Lemma 6.7 applied to u in $B_{r_k}(\pm p_k)$, where $p_k = \rho_{3r_k}\big((3r_k/2, 0)\big)$, since the zero phase of u in $B_{3r_k} \supseteq B_{r_k}(\pm p_k)$ is contained in a $(3\delta r_k)$-strip that disconnects $B_{r_k}(\pm p_k)^+(u)$ into two components. An a result, if we represent the rotation ρ_{r_k} as a complex phase $e^{i\theta_k}$, we must have
$$|e^{i\theta_{k+1}} - e^{i\theta_k}| \leq c\delta, \tag{20}$$
for the Lipschitz graph pieces $F(u) \cap \overline{B_{r_{k+1}} \setminus B_{r_k}}$ to be appropriately aligned in successive dyadic annuli.

We shall carry out the proof of the lemma in a couple of steps.

Step 1. Define $A_k := \Omega_N \cap \overline{B_{r_{k+1}} \setminus B_{r_k}}$. We shall show that
$$|U(e^{i\theta_k}\zeta) - \zeta| \leq C\delta|\zeta| \quad \text{for} \quad \zeta \in \tilde{A}_k := e^{-i\theta_k} A_k \qquad k = 2, 3, \ldots, k_0 - 1. \tag{21}$$
Define $\tilde{U}(\zeta) := U(e^{i\theta_k}\zeta)$ and let $\tilde{u} := \mathrm{Im}(\tilde{U})$. First, claim that
$$|\tilde{U}'(\zeta) - 1| \leq c\delta \quad \text{for } \zeta \in \tilde{A}_k. \tag{22}$$

The Cauchy-Riemann equations say

$$\tilde{U}'(\zeta) = i\partial_{y_1}\tilde{u} + \partial_{y_2}\tilde{u}, \qquad \zeta = y_1 + iy_2,$$

so it suffices to show that

$$\nabla_y \tilde{u} = e_2 + O(\delta) \quad \text{in} \quad \tilde{A}_k,$$

where e_2 is the unit vector in the direction of y_2. This is a straightforward corollary of

$$|\tilde{u} - y_2^+| < 3\delta r_k \quad \text{in} \quad e^{-i\theta_k}\big(\Omega_N \cap (B_{3r_k} \setminus B_{r_k/2})\big) \supseteq \tilde{A}_k.$$

and the fact that $F(\tilde{u}) \cap (B_{3r_k} \setminus B_{r_k/2})$ consists of two graphs of Lipschitz norm at most $c'\delta$.

Going back to the complex coordinate $z = e^{i\theta_k}\zeta$, we see that (22) becomes

$$|U'(z) - e^{-i\theta_k}| = |U'(z)e^{i\theta_k} - 1| \leq c\delta \quad \text{for} \quad z \in A_k. \tag{23}$$

Let z_k be defined as the unique intersection point between ∂B_{r_k} and F_R^N for $k = 1, 2, \ldots, k_0$, as above. Since there is a piece-wise smooth curve $\gamma(z_k, z) \subseteq A_k$ of length $O(r_k)$ connecting z_k to any other point $z \in A_k$, integrating $(U'(s) - e^{-i\theta_k})$ along $\gamma(z_k, z)$, we obtain using (23)

$$|U(z) - e^{-i\theta_k}z - (U(z_k) - e^{-i\theta_k}z_k)| \leq C'\delta r_k \quad z \in A_k. \tag{24}$$

In order to establish (21), it suffices therefore to show that for some large enough numerical constant $\tilde{c}$

$$|U(z_k) - e^{-i\theta_k}z_k| \leq \tilde{c}r_k, \quad k = 2, 3, \ldots, k_0 - 1.$$

We shall use induction. Without of loss of generality, the complex coordinate z is chosen so that $\theta_2 = 0$. Then, since $z_2 \in \alpha_{R,\delta r_2}$,

$$|U(z_2) - e^{-i\theta_2}z_2| = |-v(z_2) - z_2| = ||z_2| - z_2|| \leq 2\delta r_2.$$

Assume the statement is true for k. Applying (24) for $z = z_{k+1} \in A_k$

$$|U(z_{k+1}) - e^{-i\theta_k}z_{k+1}| \leq C'\delta r_k + |U(z_k) - e^{-i\theta_k}z_k| \leq (C' + \tilde{c})\delta r_k.$$

Taking into account (20), we see that

$$|U(z_{k+1}) - e^{-i\theta_{k+1}}z_{k+1}| \leq (C' + \tilde{c})\delta r_k + |e^{-i\theta_{k+1}} - e^{-i\theta_k}||z_{k+1}| \leq (C'/2 + \tilde{c}/2 + c)\delta r_{k+1}.$$

and the induction step is complete once we pick $\tilde{c} = \max\{2, C' + 2c)\}$.

Step 2. We are now ready to show that U is injective on $\overline{\Omega_N \setminus B_{r_2}}$. Let $w_1, w_2 \in \overline{\Omega_N \setminus B_{r_2}}$ be such that $U(w_1) = U(w_2)$; without loss of generality $|w_1| \leq |w_2|$. Because of (21), we have

$$|U(w_1)| \leq (1 + C\delta)|w_1| \quad \text{while} \quad |U(w_2)| \geq (1 - C\delta)|w_2|.$$

Hence,

$$1 \leq \frac{|w_2|}{|w_1|} \leq \frac{|U(w_2)|/(1 - C\delta)}{|U(w_1)|/(1 + C\delta)} = \frac{1 + C\delta}{1 - C\delta} < 2.$$

so it has to be the case that both w_1, w_2 belong to $A_{k-1} \cup A_k$ for some k. Because of (23) and (20), we have

$$|U'(z) - e^{-i\theta_k}| \leq c'\delta \quad \text{for} \quad z \in A_{k-1} \cup A_k.$$

Let $\gamma(w_1, w_2)$ be a piece-wise smooth curve in $D_k := A_{k-1} \cup A_k$ connecting w_1 to w_2. It is not hard to see that, because ∂D_k can be locally represented as a graph of a Lipschitz function with Lipschitz norm bounded by some universal constant L, $\gamma(w_1, w_2)$ can be taken such that

$$\mathcal{H}^1(\gamma(w_1, w_2)) \leq \sqrt{1 + L^2}|w_1 - w_2|.$$

Then

$$0 = U(w_2) - U(w_1) = \int_{\gamma(w_1,w_2)} U'(z)dz = e^{-i\theta_k}(w_2 - w_1) + \int_{\gamma(w_1,w_2)} (U'(z) - e^{-i\theta_k})dz,$$

so that

$$|w_1 - w_2| = \left|\int_{\gamma(w_1,w_2)} (U'(z) - e^{-i\theta_k})dz\right| \leq c'\delta\mathcal{H}^1(\gamma(w_1, w_2)) \leq c'\sqrt{1 + L^2}\delta|w_1 - w_2|,$$

which implies that $w_1 = w_2$ when δ is small enough.

Step 3. Finally, we see that (19) follows from (21) and the fact that $\mathrm{Im}(U) = u \geq 0$ with $\mathrm{Im}(U)(z) = 0$ precisely when $z \in F(u) \cap \overline{\Omega_N \setminus B_{r_2}}$. □

Lemma 9.3. *The two curves $F(u) \cap \overline{\Omega_N \setminus B_{r_3}}$ are graphs over the line $\rho_{r_{k_0}}\{y_2 = 0\}$ with Lipschitz norm at most $c\delta$ for some numerical constant c.*

Proof. From Lemma 9.2 we know that the inverse of U is well defined on the annulus

$$A := \{\xi \in \mathbb{C} : \mathrm{Im}(\xi) \geq 0, r_2(1 + C\delta) \leq |\xi| \leq r_{k_0}(1 - C\delta)\}.$$

Then $U^{-1} \circ \exp$ maps the strip

$$S = \{z \in \mathbb{C} : 0 \leq \mathrm{Im}(z) \leq \pi, \log\big(r_2(1 + C\delta)\big) \leq \mathrm{Re}(z) \leq \log\big(r_{k_0}(1 - C\delta)\big)\}.$$

conformally onto its image in $\overline{\Omega_N \setminus B_{r_2}}$: with $S \cap \{\mathrm{Im}(z) = 0\})$ parameterizing a subarc of the "right" strand F_R^N, and $S \cap \{\mathrm{Im}(z) = \pi\})$ parameterizing a subarc of the "left" strand F_L^N, under $U^{-1} \circ \exp$ (see the discussion at the beginning of the section for definitions). Since $U' \neq 0$ on $\Omega_N \setminus B_{r_2}$ and $\Omega_N \setminus B_{r_2}$ is simply-connected, one may define a branch of its logarithm $\log U'$. Finally, define the holomorphic function $\mathcal{F} : S \to \mathbb{C}$ via:

$$\mathcal{F} := \log U' \circ U^{-1} \circ \exp,$$

and let

$$f = \mathrm{Re}(\mathcal{F}) \quad \text{and} \quad g = \mathrm{Im}(\mathcal{F}).$$

Since for $\zeta_1, \zeta_2 \in F(u)$, $\big|\mathrm{Im}\big(\log U'(\zeta_2) - \log U'(\zeta_1)\big)\big|$ measures the angle of turning of ∇u along $F(u)$ from ζ_1 to ζ_2, we are going to be interested in estimating the oscillations

$$\omega_{g,L} := \mathrm{osc}\{g(z) : z \in \tilde{S} \cap \{\mathrm{Im}(z) = \pi\}\} \qquad \omega_{g,R} := \mathrm{osc}\{g(z) : z \in \tilde{S} \cap \{\mathrm{Im}(z) = 0\}\}.$$

where

$$\tilde{S} = \{z \in \mathbb{C} : 0 \leq \mathrm{Im}(z) \leq \pi,\ \log r_2 + c_0 \leq \mathrm{Re}(z) \leq \log r_{k_0} - c_0\} \subseteq S \qquad c_0 = \log 2.$$

We would like to show that both $\omega_{g,L}$ and $\omega_{g,R} = O(\delta)$, as this would imply that the amount of turning of ∇u along F_L^N (F_R^N), from ∂B_{2r_2} to $\partial B_{r_{k_0}/2}$, is $O(\delta)$, which, in turn, would be enough to conclude that $F_L^N \cap \overline{B_{r_{k_0}} \setminus B_{r_3}}$ and $F_R^N \cap \overline{B_{r_{k_0}} \setminus B_{r_3}}$ are in fact graphs of Lipschitz constant $O(\delta)$ (as we already know that $F_L^N \cap \overline{B_{r_{k_0}} \setminus B_{r_{k_0}/2}}$ and $F_R^N \cap \overline{B_{r_{k_0}} \setminus B_{r_{k_0}/2}}$ are graphs of Lipschitz constant $O(\delta)$). To that goal we would like to obtain estimates on $|\nabla g|$ in ∂S, which by the Cauchy–Riemann equations satisfies

$$|\nabla g| = |\nabla f| \quad \text{in } S.$$

For convenience, define the following coordinates on S

$$t = \mathrm{Re}(z) - (\log\big(r_2(1 + C\delta)\big) + A) \qquad \theta = \mathrm{Im}(z) - \pi/2, \quad \text{where}$$

$$2A = \log\big(r_{k_0}(1 - C\delta)\big) - \log\big(r_2(1 + C\delta)\big) = \log\frac{1 - C\delta}{1 + C\delta} + (k_0 - 2)\log 2 = (k_0 - 2)\log 2 + O(\delta)$$

by translating the coordinates $(\mathrm{Re}(z), \mathrm{Im}(z))$ appropriately, so that S is parameterized by

$$S = \{|t| \leq A, |\theta| \leq \pi/2\}.$$

Note that since $|U'| = |\nabla u|$ on $F(u) \cap (\Omega_N \setminus B_{r_2})$, we have

$$|f(t, \pm\pi/2)| = \log|\mathcal{F}| = \log 1 = 0 \quad \text{for} \quad |t| \leq A.$$

Also, by the estimate (22) of Lemma 9.2, we have

$$|f(\pm A, \theta)| \leq c\delta \quad \text{for} \quad |\theta| \leq \pi/2.$$

Applying Schwarz reflection across $\theta = -\pi/2$ and $\theta = \pi/2$, we can extend f to a harmonic function on $\hat{S} := \{|t| \leq A, |\theta| \leq 3\pi/2\}$. By the maximum principle $|f| \leq c\delta$ in $\hat{S}$. Denote $\tilde{A} := A - c_0/2$. Interior estimates for f then yield

$$|\nabla f(\pm\tilde{A}, \theta)| \leq \tilde{c}\|f\|_{L^\infty(\hat{S})} \leq \tilde{C}\delta \quad \text{for} \quad |\theta| \leq \pi/2, \tag{25}$$

which we can integrate to get

$$|f(\pm\tilde{A}, \theta)| \leq \tilde{C}\delta\cos\theta \quad \text{for} \quad |\theta| \leq \pi/2.$$

Using multiples of $\cosh t \cos\theta$ as upper and lower barriers for f, we have the bound

$$-(c\delta/\cosh\tilde{A})\cosh t\cos\theta \leq f \leq (c\delta/\cosh\tilde{A})\cosh t\cos\theta \quad \text{in} \quad \tilde{S},$$

so that, by the Hopf Lemma, we can conclude

$$|\nabla f(t, \pm\pi/2)| = |\partial_\theta f(t, \pm\pi/2)| \leq (c\delta/\cosh \tilde{A}) \cosh t \quad \text{for} \quad |t| \leq \tilde{A}. \tag{26}$$

This in turn implies the desired

$$\omega_{g,L} \leq \int_{-\tilde{A}}^{\tilde{A}} |\nabla g(t, \pi/2)| \, dt \leq 2c\delta, \qquad \omega_{g,R} \leq \int_{-\tilde{A}}^{\tilde{A}} |\nabla g(t, -\pi/2)| \, dt \leq 2c\delta.$$

□

10. Curvature bounds.

Let us describe the family of hairpin solutions explicitly. Define

$$\varphi(\zeta) = i(\zeta + \sinh \zeta).$$

Then φ maps the strip $S = \{|\mathrm{Im}\zeta| < \pi/2\}$ conformally onto the domain

$$\Omega_1 := \{z \in \mathbb{C} : |\mathrm{Re} z| < \pi/2 + \cosh(\mathrm{Im} z)\}$$

which supports the positive phase of the hairpin solution of (1)

$$H(z) = \begin{cases} \mathrm{Re}\big(V(z)\big) & \text{when } z \in \Omega_1 \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

where $V(z) := \cosh(\varphi^{-1}(z))$. The dilates

$$H_a(z) = aH(z/a)$$

complete the family of hairpin solution (up to rigid motions). Denote by $\Omega_a = a\Omega_1$.

We note a couple of geometric features of these solutions.

- The zero phase $\{H_a = 0\}$ consists of two connected components separated by distance $s = a(2 + \pi)$.
- $H_a\big|_{\Omega_a}$ has a unique critical point (a non-degenerate saddle) and it is situated at the origin. Indeed, to verify this, we have to simply check this is the obviously the case for
$$H(\varphi(\zeta)) = \mathrm{Re} \cosh(\zeta) = \cosh(y_1)\cos(y_2) \quad \text{when} \quad \zeta = y_1 + iy_2 \in S.$$
The value of H_a at the saddle is precisely $H_a(0) = a$.
- The segments $\tau_{a,L} := [-s/2, 0] \subseteq \mathbb{C}$ and $\tau_{a,R} := [0, s/2] \subseteq \mathbb{C}$ are the steepest descent paths from 0 to each of the two components of $\{H_a = 0\}$, respectively. We shall denote $\tau_a := \tau_{a,L} \cup \tau_{a,R}$.

The following information about the gradient ∇H will also be useful.

Lemma 10.1. *For some numerical constant $c_0 > 0$, the gradient ∇H satisfies*

$$|\nabla H(x)| \geq \min(1/2, c_0|x|) \quad \textit{when} \quad x \in \Omega_1.$$

Proof. We have

$$|\nabla H|(\varphi(\zeta)) = \left|\frac{i \sinh \zeta}{\varphi'(\zeta)}\right| = \sqrt{\frac{\sinh^2 y_1 + \sin^2 y_2}{(1 + \cosh y_1 \cos y_2)^2 + \sinh^2 y_1 \sin^2 y_2}} = \frac{\sqrt{\sinh^2 y_1 + \sin^2 y_2}}{\cosh y_1 + \cos y_2} \quad \zeta = y_1 + iy_2 \in S.$$

Thus,

$$|\nabla H|(\varphi(\zeta)) \geq \frac{|\sinh y_1|}{1 + \cosh y_1} = |\tanh(y_1/2)| \geq 1/2 \quad \text{when} \quad |y_1| \geq 1.2$$

We'll be done once we show that

$$|\nabla H|(\varphi(\zeta)) \geq c_0|\varphi(\zeta)| \quad \text{when } |\mathrm{Re}\zeta| = |y_1| < 1.2.$$

Since for some numerical constants $0 < c_1 < c_2$

$$c_1|\zeta| \leq |\sinh \zeta| \leq c_2|\zeta| \quad \text{when } |\mathrm{Re}\zeta| = |y_1| < 1.2$$

we have for $|\mathrm{Re}\zeta| < 1.2$,

$$|\nabla H|(\varphi(\zeta)) \geq \frac{|\sinh \zeta|}{\cosh y_1 + \cos y_2} \geq \frac{c_1|\zeta|}{\cosh(1.2) + 1} \geq \tilde{c}_1|\zeta|.$$

Noting that $|\varphi(\zeta)| \leq |\zeta| + |\sinh z| \leq (1 + c_2)|\zeta|$ when $|\mathrm{Re}\zeta| < 1.2$, we complete the proof of the lemma. □

Remark 10.2. *Finally, we would like to make the following remark regarding the mapping properties of* $V_a(z) := aV(z/a)$ *on* Ω_a. *Claim that* V_a *maps both* $\Omega_a^+ := \Omega_a \cap \{x_2 > 0\}$ *and* $\Omega_a^- := \Omega_a \cap \{x_2 < 0\}$ *conformally onto*

$$\tilde{\mathbb{H}}_a := \{\xi : Re(\xi) > 0\} \setminus (0, a].$$

Indeed, let $S_\pm = \{\zeta \in \mathbb{C} : \pm Re\ \zeta > 0, |Im\ \zeta| < \pi/2\}$. *Then* $(a\varphi)$ *is a conformal map from* $S_\pm$ *onto* $\Omega_a^\pm$ *and*

$$V_a(a\varphi(\zeta)) = a\cosh(\zeta) = a\cosh y_1 \cos y_2 + ia \sinh y_1 \sin y_2 \quad \text{when} \quad \zeta = y_1 + iy_2 \in S_\pm.$$

Write

$$V_a(a\phi(\zeta)) = r(\zeta)e^{i\theta(z)}$$

where

$$r(\zeta)^2/a^2 = |V_a\phi(\zeta))|^2 = \sinh^2 y_1 + \cos^2 y_2 \qquad \tan\theta(\zeta) = \tanh y_1 \tan y_2.$$

If θ_0 *is any angle in* $(-\pi/2, 0) \cap (0, \pi/2)$ *and* $c := \tan\theta_0$, *then* $\tan\theta(\zeta) = c$ *whenever* $\tan y_2 = c \coth y_1$, *so that for these values of* ζ,

$$r(\zeta)^2/a^2 = \sinh^2 y_1 + 1/(\tan^2 y_2 + 1) = \sinh^2 y_1 + 1/(1 + c^2\coth^2(y_1)).$$

We see that $r(\zeta) \to 0$ *as* $y_1 \to 0$ *and* $r(\zeta) \to \infty$ *as* $y_1 \to \pm\infty$, *so* $V_a(a\varphi)|_{S^\pm}$ *is onto* $\mathbb{H}_\infty$. *When* $\theta_0 = 0$, *i.e.* $Im(V_a(a\varphi)) = 0$, *so it must be that* $y_2 = 0$ *and thus,* $r(\zeta) = a\cosh y_1$ *which ranges in* (a, ∞). *Hence,* $V_a(a\varphi)|_{S^\pm}$ *is surjective onto* $\mathbb{H}_a$. *To show that say* $V_a(a\varphi)|_{S^+}$ *is injective, assume that for some* $y_1 + iy_2, \tilde{y}_1 + i\tilde{y_2} \in S^+$ *with* $y_1 \le \tilde{y}_1$,

$$\cosh y_1 \cos y_2 = \cosh \tilde{y}_1 \cos \tilde{y}_2 \quad \text{and} \quad \sinh y_1 \sin y_2 = \sinh \tilde{y}_1 \sin \tilde{y}_2.$$

Divide the second equation by the first to get

$$\tanh y_1 \tan y_2 = \tanh \tilde{y}_1 \tan \tilde{y}_2.$$

We see that if $y_1 = \tilde{y_1}$ *we must have* $y_2 = \tilde{y}_2$ *too. Assume* $y_1 < \tilde{y}_1$; *then either* $y_2 = \tilde{y}_2 = 0$, *which then implies* $\cosh y_1 = \cosh \tilde{y_1}$ *contradicting the assumption* $0 < y_1 < \tilde{y}_1$, *or* $|\tan y_2| > |\tan \tilde{y}_2|$ *which implies* $\cos y_2 < \cos \tilde{y}_2$. *But in the latter case,*

$$\cosh y_1 \cos y_2 < \cosh \tilde{y}_1 \cos \tilde{y}_2,$$

so we get a contradiction again. Similarly, we show the injectivity of $V_a(a\varphi)|_{S^-}$.

Having amassed enough information about the model hairpin solutions let us explore how well they approximate classical solutions of (1) whose zero phase has two connected components that are sufficiently close to each other.

Proposition 10.3. *Let* u *be a classical solution of* (1) *in* B_1 *that satisfies* (3). *Assume that* $\{u = 0\}$ *consists of two connected components* Z_L *and* Z_R *and that* 0 *is the midpoint of a shortest segment between* Z_L *and* Z_R. *For any given* $\delta_1 > 0$ *and every* $M > 0$, *there exists* $s_1 > 0$ *such that if*

$$s := d(Z_L, Z_R) \le s_1,$$

then after a rotation

$$|u(ax)/a - H(x)| \le \delta_1 \quad \text{for all} \quad |x| \le M, \tag{28}$$

where $a = s/(2 + \pi)$.

Proof. Fix δ_1 and M and assume no such s_1 exists that makes the proposition valid. Then for some sequence of $s_k \to 0$, there is a sequence of counterexamples u_k with the separation between the two components of $\{u_=0\}$ being s_k. Set $a_k = s_k/(2 + \pi)$. We can then define the rescales

$$\tilde{u}_k(x) := u(a_k x)/a_k \quad \text{for } x \in B_{1/s_k}$$

so that the separation between the two components of the zero phase of $\tilde{u}_k$ is precisely $(2 + \pi)$ and 0 is at the midpoint of a shortest segment connecting them. A subsequence u_{k_j} converges uniformly on $\overline{B_M}$ to a global solution $\tilde{u}$ and since

$$d_H((\overline{B_M})^+(\tilde{u}_{k_j}), (\overline{B_M})^+(\tilde{u})) \to 0 \quad \text{as } j \to \infty,$$

it has to be the case that $\tilde{u}$ is a hairpin solution, with separation between the two components of $\{\tilde{u} = 0\}$ precisely $(2 + \pi)$ and 0 at the midpoint of the shortest segment. Thus, in a rotated coordinate system

$$\tilde{u} = H,$$

so we have for all j large enough

$$|u_{k_j}(a_{k_j}x)/a_{k_j} - H(x)| \le \delta_1 \quad \text{in } \overline{B_M}.$$

This contradicts the assumption that the u_{k_j} are counterexamples. □

The next corollary is a direct consequence of interior estimates applied to the proposition above.

Corollary 10.4. *Let u be as in Proposition 10.3 and let H be the hairpin solution defined above. For every $M > 0$, any compact domains K, K' such that $K \Subset K' \Subset (\overline{B_M})^+(H)$ and any $\delta_1 > 0$ such that $\mathcal{N}_{2\delta_1}(K') \subseteq (\overline{B_M})^+(H)$, there exists $s_1 > 0$, such that if the separation between the two components Z_L and Z_R of $\{u = 0\}$ satisfies*

$$s := d(Z_L, Z_R) \le s_1,$$

then for $a = s/(2+\pi)$, in some rotated coordinate system, the rescale $u_a := u(ax)/a$ satisfies

$$\|u_a - H\|_{C^2(K)} \le C_{K,K'}\delta_1 \tag{29}$$

for some constant $C_{K,K'}$, dependent on K and K'. Furthermore, if $\delta_1 = \delta_1(H)$ is small enough, u has a unique critical point x_0 in $B^+_{aM/2}(u)$ which is a non-degenerate saddle point with $|x_0| = O(\delta_1 s)$, and the steepest descent paths β_L, β_R for u from x_0 to Z_L and Z_R, respectively, are contained in an $O(\delta_1 s)$-neighborhood of τ_a (defined above).

Proof. Let s_1 be such that according to (28)

$$|u_a - H(x)| \le c\delta_1 \quad \text{for all} \quad |x| \le M,$$

where we pick c such that $H(x) \ge cd(x, F(H))$ (such a c exists because of the non-degeneracy of the hairpin solution). Then for all $x \in K'$

$$u_a(x) \ge H(x) - c\delta_1 \ge cd(x, F(H)) - c\delta_1 > 2c\delta_1 - c\delta_1 > 0.$$

Hence $v = u_a - H$ is harmonic in K' and we get (29) by standard interior estimates.

Let us use this to show that u has a unique critical point in $B^+_{aM/2}(u)$ if δ_1 is small enough. Fix $\delta_0 > 0$ small and find a scale $r_0 = r_0(\delta_0, H)$ such that for every $p \in F(H) \cap \overline{B_{M/2}}$

$$d_H(F(H) \cap B_{r_0}(p), L(p) \cap B_{r_0}(p)) < \delta_0 r_0,$$

where $L(p)$ denote the straight line tangent to $F(H)$ at p. Now, for all small enough $\delta_1 < \delta_0 r_0$

$$d_H(F(u_a) \cap B_{r_0}(p), L(p) \cap B_{r_0}(p)) < c'\delta_0 r_0.$$

Hence, $F(u_a)$ is $c'\delta_0$-flat in $B_{r_0}(p)$ and it must be that for a small enough δ_0,

$$|\nabla u_a - \nabla H(p)| \le C\delta_0 \quad \text{in} \quad B_{r_0/2}(p)^+(u_a) \quad \text{for any } p \in F(H) \cap \overline{B_{M/2}} \tag{30}$$

by the classical Alt-Caffarelli theory [AC81]. Thus,

$$|\nabla u_a| \ge 1 - C\delta_0 \quad \text{in} \quad B_{r_0/2}(p)^+(u_a)$$

and so it suffices to show that u_a has a unique critical point in

$$K := \{x \in B^+_{M/2}(H) : d(x, \partial\Omega_1\} \ge r_0/4\}.$$

Set $K' := \{x \in B^+_{2M/3}(H) : d(x, \partial\Omega_1\} \ge r_0/8\}$. We would like to show that

$$0 = \nabla u_a = \nabla H + \nabla v$$

has a unique solution $x_{0,a}$ in K. Since the Jacobian of $\nabla H = D^2 H$ is invertible at 0, the Inverse Function Theorem implies that for some $c_1 > 0$, ∇H maps B_{c_1} diffeomorphically onto a neighborhood O of 0. As $|\nabla v| \le C_{K,K'}\delta_1$ in K, we can choose δ_1 small enough such that $\nabla v \in O$, whence

$$\nabla H(x) = -\nabla v(x)$$

has a unique solution $x = x_{0,a} \in B_{c_1}$. Applying Lemma 10.1, we obtain

$$|x_{0,a}| \le c_0^{-1}|\nabla H(x_{0,a})| = c_0^{-1}|\nabla v(x_{0,a})| \le c_0^{-1}C_{K,K'}\delta_1.$$

Furthermore, the equation cannot have another solution in K if δ_1 is small enough, because Lemma 10.1 implies that

$$|\nabla H(x)| \ge \min(1/2, cc_1) > C_{K,K'}\delta_1 \ge |\nabla v(x)| \quad \text{for all} \quad x \in K \setminus B_{c_1}.$$

Thus, whenever δ_1 is small enough, u_a has a unique critical point $x_{0,a}$ in K and since

$$|D^2u_a(x_{0,a}) - D^2H(0)| \leq |D^2u_a(x_{0,a}) - D^2H(x_{0,a})| + |D^2H(x_{0,a}) - D^2H(0)| \\ = O(\delta_1) + O(|x_{0,a}|) = O(\delta_1),$$

$x_{0,a}$ is a non-degenerate saddle point.

The $O(\delta_1)$ proximity between the steepest descent paths for u_a and H from $x_{0,a}$ and 0, respectively, to their zero sets, follows from the $O(\delta_1)$ bound for $\nabla(u_a - H)$ in K and (30). □

Let $\delta_0 > 0$ be small constant from Remark 8.1. Let us present the set-up that we shall be working in for the rest of the section. The object of interest is

(1) A classical solution u of (1) in B_1 that satisfies (3) such that $\{u = 0\}$ consists of two connected components Z_L and Z_R with 0 being at the midpoint of the shortest segment between the two.
(2) The free boundary $F(u)$ consists of two arcs $F_L := F(u) \cap \partial Z_L$ and $F_R = F(u) \cap \partial Z_R$.
(3) We assume that $\delta_1 \leq \delta < \delta_0$, M, s_1, s are as in Proposition 10.3 and Corollary 10.4, i.e. the fact that $d(Z_L, Z_R) = s < s_1$ implies

$$|u - H_a| \leq \delta_1 a \quad \text{in} \quad B_{aM} \quad \text{where} \quad a = (2+\pi)s$$

and u has a unique critical point x_0 in $B^+_{aM/2}(u)$. We denote by β_L be the steepest descent path for u that connects x_0 to some $p \in F_L$ and by β_R be the steepest descent path from x_0 to some $q \in F_R$. Then $\beta := \beta_L \cup \beta_R$ is a smooth arc connecting F_L to F_R and $\beta \subseteq \mathcal{N}_{a\delta_1}(\tau_a)$. Without loss of generality, we may assume that our coordinate system is chosen in such a way that

$$\nabla u(p) = e_1.$$

(4) Furthermore, for some rotation ρ

$$|u(\rho x) - |x_2|| \leq \delta \quad \text{in all of} \quad B_1.$$

and $F(u) \cap (B_{2/3} \setminus B_{4Ms})$ consists of four graphs over $\rho(\{x_2 = 0\})$ of Lipschitz norm at most $C\delta$.

Let $\partial B_{1/2}$ intersect F_L at the two points p_N and p_S (subscripts N and S are determined by $x_2(\rho^{-1}(p_N)) > x_2(\rho^{-1}(p_S))$) and similarly $\partial B_{1/2}$ intersects F_R at the two points q_N and q_S. Define $\Omega_N \subseteq B^+_{1/2}(u)$ to be the domain bounded by the subarc of F_L from p_N to p, the arc β, the subarc of F_R from q to q_N and by the circular arc of $\partial B_{1/2}$ with ends p_N and q_N, which contains $\rho(0, 1/2)$. Analogously, define Ω_S to be the domain bounded by subarc of F_L from p_S to p, the arc β, the subarc of F_R from q to q_S and by the circular arc of $\partial B_{1/2}$ with ends p_S and q_S, which contains $\rho(0, -1/2)$. Then

$$B^+_{1/2}(u) = \Omega_N \sqcup \beta \sqcup \Omega_S.$$

Let v be the harmonic conjugate of u in the simply-connected $B^+_1(u)$ where we choose the normalization

$$v(x_0) = 0.$$

Note that this implies $v = 0$ on all of β, as ∇v is a rotation by $\pi/2$ of ∇u which itself is tangent to β. Furthermore v is increasing (decreasing) at unit speed along $F_L \cap \Omega_N$ ($F_L \cap \Omega_S$) and decreasing (increasing) at unit speed along $F_R \cap \Omega_N$ ($F_R \cap \Omega_S$) as we move towards $\partial B_{1/2}$.

Define the holomorphic map $U : B^+_1(u) \to \mathbb{C}$ via

$$U = u + iv.$$

The next lemma confirms that the mapping properties of U are similar to those of V_a (defined in Remark 10.2), which allows us to construct an injective holomorphic map from $B^+_{1/2}(u)$ to Ω_{a_0} for some $a_0 > 0$.

Lemma 10.5. *Provided δ and δ_1 small enough, U is injective on each of Ω_N and Ω_S and maps each of β_L and β_R injectively onto $i[0, a_0]$, where $a_0 := u(x_0) = a(1 + O(\delta_1))$. Then the map $\tilde{\psi} : B^+_{1/2}(u) \setminus \beta \to \Omega_{a_0}$ given by*

$$\tilde{\psi}(z) := \begin{array}{ll} \left(V_{a_0}|_{\Omega^+_{a_0}}\right)^{-1} \circ U(z) & \text{when} \quad z \in \Omega_N \\ \left(V_{a_0}|_{\Omega^-_{a_0}}\right)^{-1} \circ U(z) & \text{when} \quad z \in \Omega_S \end{array}$$

is injective and in fact extends continuously to β. The extension $\psi : B^+_{1/2}(u) \to \Omega_{a_0}$ defines, therefore, an injective holomorphic map whose image contains

$$\psi(B^+_{1/2}(u)) \supseteq \Omega_{a_0} \cap B_{1/4}$$

Proof. Let us first show that U is injective in Ω_N. Since U maps each of β_L and β_R injectively onto $[0, a_0]$ and since near x_0

$$U(z) = a_0 + c(z - x_0)^2 + O(|z - x_0|^3)$$

by the smoothness of U, for every small enough $\epsilon > 0$ we can find an arc $\beta_\epsilon \subseteq \Omega_N \cap \mathcal{N}_\epsilon(\beta)$ connecting $p_\epsilon \in F_L$ to $q_\epsilon \in F_R$, such that U maps it bijectively onto an arc $\gamma_\epsilon \subseteq \mathbb{H}_a$ with ends $U(p_\epsilon)$ and $U(q_\epsilon)$, where

$$\mathrm{Re}(U(p_\epsilon)) = \mathrm{Re}(U(q_\epsilon)) = 0 \quad \text{and} \quad \mathrm{Im}(U(p_\epsilon)) = v(p_\epsilon) > 0 > v(q_\epsilon) = \mathrm{Im}(U(q_\epsilon))$$

Let $\Omega_{N,\epsilon}$ be the domain bounded by the subarc of F_L with ends p_N and p_ϵ, β_ϵ, the subarc of F_R with ends q_ϵ and q_N, and the corresponding circular arc $\widehat{p_N q_N}$ of $\partial B_{1/2}$. Claim that U is injective on the closed Jordan arc $\partial\Omega_{N,\epsilon}$. We can easily see that U maps $(F(u) \cap \partial\Omega_{N,\epsilon}) \cup \beta_\epsilon$ injectively onto

$$\Gamma_\epsilon := \gamma_\epsilon \cup \{y_1 = 0, y_2 \in [-l_L, l_R]\} \setminus \{y_1 = 0, y_2 \in (v(q_\epsilon), v(p_\epsilon))\}$$

where

$$l_L := \mathcal{H}^1(\partial\Omega_N \cap F_L) \geq 2/5 \quad \text{and} \quad l_R := \mathcal{H}^1(\partial\Omega_N \cap F_R) \geq 2/5.$$

It remains to confirm that U is injective on $\widehat{p_N q_N}$ and that $U(\widehat{p_N q_N}) \cap \Gamma_\epsilon = U(\widehat{p_N q_N}) \cap \gamma_\epsilon = \emptyset$. Those follow easily from the fact that

$$|U'(z)e^{i\theta} - (-i)| \leq c\delta \quad z \in \widehat{p_N q_N} \tag{31}$$

where $e^{i\theta}$ represents the rotation ρ.

Since U is injective on $\partial\Omega_{N,\epsilon}$, $U(\partial\Omega_{N,\epsilon})$ is a closed Jordan arc that divides $\mathbb{C}$ into a bounded domain D_b and an unbounded domain D_u. For $\xi_0 \notin U(\partial\Omega_{N,\epsilon})$,

$$Q(\xi_0) := \frac{1}{2\pi i} \oint_{\partial\Omega_{N,\epsilon}} \frac{dz}{U(z) - \xi_0} = \frac{1}{2\pi i} \oint_{U(\partial\Omega_{N,\epsilon})} \frac{d\xi}{\xi - \xi_0}$$

equals the winding number of the closed Jordan arc $U(\partial\Omega_{N,\epsilon})$ around ξ_0, i.e. $Q(\xi_0) = 1$ when $\xi_0 \in D_b$ and $Q(\xi_0) = 0$ when $\xi \in D_u$. On the other hand, by the Argument Principle, $Q(\xi_0)$ equals the number of zeros (with multiplicities) of $U(z) = \xi_0$ in $\Omega_{N,\epsilon}$. We can thus conclude that U is injective on $\Omega_{N,\epsilon}$.

Taking a sequence $\epsilon_k \to 0$ we construct a sequence of domains Ω_{N,ϵ_k} such that $\Omega_N = \bigcup_k \Omega_{N,\epsilon_k}$ with U injective on each Ω_{N,ϵ_k}. Therefore, U is injective on all of Ω_N. Analogously, we establish the injectivity of U on Ω_S.

Finally, let's show that ψ extends continuously to β. Let z belong to the interior of the arc β_L, and let $\{z_{N,k}\} \subseteq \Omega_N$, $\{z_{S,k}\} \subseteq \Omega_S$ be two sequences such that both

$$z_{N,k} \to z \quad z_{S,k} \to z.$$

Denote $\xi_{N,k} := U(z_{N,k})$ and $\xi_{S,k} := U(z_{S,k})$. Then we see that both

$$\xi_{N,k} \to u(z) + i0^+ \quad \text{and} \quad \xi_{S,k} \to u(z) + i0^+$$

with $u(z) \in (0, a_0)$. Then if $\zeta_{N,k} = V_{a_0}^{-1}|_{\Omega^+_{a_0}}(\xi_{N,k})$ and $\zeta_{S,k} = V_{a_0}^{-1}|_{\Omega^-_{a_0}}(\xi_{N,k})$, we can easily verify that

$$\zeta_{N,k} \to b + i0^+ \quad \text{and} \quad \zeta_{N,k} \to b + i0^-$$

with $b \in \tau_{a_0,L}$ being the unique point of $\tau_{a_0,L}$ that V_{a_0} maps to $u(z) \in (0, a_0)$. Hence, $\tilde{\psi}$ can be continuously extended on the interior of β_L and similarly, onto the interior of β_R. Since this extension is bounded in the vicinity of x_0, it further extends to a holomorphic function ψ in all of $B^+_{1/2}(u)$ with $\psi(x_0) = 0$. Since ψ maps Ω_N injectively into $\Omega^+_{a_0}$ and Ω_S injectively into $\Omega^-_{a_0}$, $(\beta_L)^\circ$ injectively into $(\tau_{a_0,L})^\circ$ and $(\beta_R)^\circ$ injectively into $(\tau_{a_0,R})^\circ$, we conclude that ψ is injective on all of $B^+_{1/2}(u)$.

Lastly, we point out that since U maps $\partial\Omega_N \cap F(u)$ onto $[-l_L, l_R]$ and maps $\partial\Omega_N \cap \partial B_{1/2}$ into a curve that is $O(\delta_1)$-close to a half-circle of radius $1/2$, according to (31), it has to be that

$$U(\Omega_N) \supseteq \mathbb{H}_{a_0} \cap B_{1/3}.$$

Thus for all small a_0, $\psi(\Omega_N) = (V_{a_0}|_{\Omega^+_{a_0}})^{-1}\big(U(\Omega_n)\big) \supseteq \Omega^+_{a_0} \cap B_{1/4}$. After applying the same argument for Ω_S, we establish the full statement $\psi(B^+_{1/2}(u)) \supseteq \Omega_{a_0} \cap B_{1/4}$.

□

We shall now use the map ψ to obtain curvature bounds of $F(u)$ in $B_{1/4}$. On the road to do so, we will obtain the following crucial estimates on ψ' and ψ''.

Lemma 10.6. *The injective holomorphic map $\psi : B^+_{1/2}(u) \to \Omega_{a_0}$ constructed in Lemma 10.5 satisfies:*

$$|\psi''(z)| \leq C\delta \quad \textit{and} \quad |\psi'(z) - 1| \leq C\delta(|z| + a_0) \quad \textit{for} \quad z \in B^+_{1/4}(u).$$

Proof. We know that for $z \in \partial B_{1/2} \cap \partial B^+_{1/2}(u)$

$$|\psi'(z)| = \frac{|U'(z)|}{|V'_{a_0}(\psi(z))|} = 1 + O(\delta)$$

because $|U'(z)| = 1 + O(\delta)$ for $\partial B_{1/2} \cap B^+_1(u)$ and

$$|V'_{a_0}(\psi(z))| = |V'(\psi(z)/a_0)| = 1 + O(\delta) \quad z \in \partial B_{1/2} \cap B^+_1(u)$$

for all $a_0 = a(1 + O(\delta_1))$ small enough (depending on δ), because according to Lemma 10.5

$$|\psi(z)| \geq 1/4 \quad \text{when} \quad z \in \partial B_{1/2} \cap B^+_1(u).$$

Furthermore, $|\psi'| = 1$ on $F(u) \cap B_{1/2}$, so that by the maximum (and minimum) modulus principle,

$$|\psi'| = 1 + O(\delta) \quad \text{in} \quad B^+_{1/2}(u). \tag{32}$$

Since $B^+_{1/2}(u)$ is simply-connected and since $\psi' \neq 0$ as ψ is conformal, we can write

$$\psi' = e^G$$

for some holomorphic function G on $B^+_{1/2}(u)$. Then

$$\psi'' = G'\psi'$$

and in view of (32), we shall have $\psi'' = O(\delta)$ in $B^+_{1/4}(u)$ once we establish

$$|G'| \leq c\delta \quad \text{in} \quad B^+_{1/4}(u).$$

Let $g = \mathrm{Re}(G)$; as $|G'| = |\nabla g|$ it suffices to obtain bounds on $|\nabla g|$ and we know that

$$g(z) = \log|\psi'(z)| = \begin{cases} 0 & z \in F(u) \cap B_{1/2} \\ O(\delta) & z \in B^+_{1/2}(u) \end{cases}$$

In particular g vanishes on $F(u) \cap B_{1/2}$ and we can apply the boundary Harnack inequality in the $C\delta$-Lipschitz domains $B_{1/4}(z_\pm)^+(u)$, where $z_\pm := \rho(\pm 1/4, 0)$, in order to establish that

$$|g(z)| \leq c\delta u(z)/u(z_\pm \pm i/8) \leq c'\delta u(z) \quad \text{in} \quad B_{1/8}(z_\pm)^+(u)$$

(since by assumption (4), $u(z_\pm \pm i/8) \approx 1/8$). Because we have

$$u \geq 1/8 - \delta \geq 1/10 \quad \text{on} \quad \partial B_{1/4} \setminus \big(B_{1/8}(z_+) \cup B_{1/8}(z_-)\big),$$

we see that $|g| \leq C\delta u$ on $\partial B_{1/4} \cap B^+_1(u)$ and thus by the maximum principle,

$$|g| \leq C\delta u \quad \text{in all of} \quad B^+_{1/4}(u).$$

An application of the Hopf Lemma yields

$$|\nabla g| \leq C\delta|\nabla u| = C\delta \quad \text{on} \quad F(u) \cap B_{1/4}.$$

Finally, we have

$$|\nabla g| = \frac{|\nabla|\psi'|^2|}{2|\psi'|^2} \leq C\delta \quad \text{on} \quad B^+_1(u) \cap \partial B_{1/4}$$

because of (32) and the fact that on $B^+_1(u) \cap \partial B_{1/4}$

$$\nabla|\psi'|^2 = 2\mathrm{Re}\left(\nabla\big(U'/(V'_{a_0} \circ \psi)\big)\overline{\psi'}\right) = O(|U''| + |V''_a \circ \psi|) = O(\delta).$$

Hence $|\nabla g| \leq C\delta$ in all of $B^+_{1/4}(u)$ as desired.

To get the first derivative bound, we integrate the second derivative bound along a curve $\gamma \subseteq B^+_{1/4}(u)$ connecting $p \in F_L \cap \beta$ (the "left" end of the steepest path) to z:

$$\psi'(z) = \psi'(x_0) + \int_\gamma \psi''(\zeta)\ d\zeta = \psi'(p) + O(\delta \mathcal{H}^1(\gamma)).$$

Since by definition $V'(\psi(p)) = U'(p) = 1$ (as $\nabla u(p) = e_1$)

$$\psi'(p) = U'(p)/V'(\psi(p)) = 1$$

As γ can be taken to be of length $O(|z| + a_0)$, we obtain the desired bound

$$|\psi'(z)| - 1| \leq C\delta(|z| + a_0) \quad z \in B^+_{1/4}(u).$$

□

Theorem 10.7. *Given $\delta > 0$ small enough, there exist $r_0 > 0$, $\epsilon_1 > 0$ such that if u is a classical solution of* (1) *in B_1, $0 \in F(u)$ and*

$$dist(0, \{u = 0\} \setminus Z) < \epsilon_1 r_0$$

then there exists a point $p \in B_{r_0/3}$ such that $B_{r_0/2}(p) \cap F(u)$ consists of two free boundary arcs F_L and F_R, the shortest segment between which is centered at p, the separation

$$s := dist(F_L, F_R) < \epsilon_1 r_0.$$

Furthermore, u has a unique saddle point x_0 in $B_{r_0/2}(p)$ and there is an injective holomorphic map

$$\psi : B_{r_0/2}(p)^+(u) \to \Omega_a \quad \textit{where} \quad a = u(x_0)$$

that extends continuously to $\partial B_{r_0/2}(p)^+(u)$, mapping $\psi(x_0) = 0$ and $F(u) \cap B_{r_0/2}(p)$ into $\partial\Omega_a$ and satisfying

$$|\psi''| \leq C\delta/r_0 \qquad |\psi' - e^{i\theta}| < C\delta(|z| + a)/r_0 \quad \textit{in} \quad B_{r_0/2}(p)^+(u) \tag{33}$$

for some $\theta \in \mathbb{R}$. It relates the curvature κ of $F(u)$ in $B_{r_0/2}(p)$ to the curvature κ_a of $\partial\Omega_{ar_0}$ via

$$|\kappa(z) - \kappa_a(\psi(z))| \leq C\delta/r_0 \quad z \in F(u) \cap B_{r_0/2}(p) \tag{34}$$

for some numerical constants $C, c > 0$.

Proof. For δ fixed we find r_0, ϵ_0 as in Theorem 9.1. Set $\delta_1 = \delta$, $M = 8/\epsilon_0$, apply Proposition 10.3 to find $s_1 = s_1(\delta, M)$ and set $\epsilon_1 = \min(\epsilon_0, s_1)$. Then $\tilde{u}(y) := (2/r_0)u(p + \rho y r_0/2)$, defined in B_1 for some appropriate rotation $\rho \sim e^{i\theta}$, falls under the set-up (1-4) and we construct the injective holomorophic map $\tilde{\psi}$ as in Lemma 10.5 satisfying the estimates of Lemma 10.6, whose rescaled statement is precisely (33).

Let $\tilde{U}$ and V_{a/r_0} be the holomorphic extensions of $\tilde{u} = \mathrm{Re}(\tilde{U})$ and $H_{2a/r_0} = \mathrm{Re}(V_{2a/r_0})$ such that

$$\tilde{U}(z) = V_{2a/r_0}(\tilde{\psi}(z)) \quad \text{in} \quad B_{1/2}.$$

Since the curvature of $F(\tilde{u})$ at z

$$\tilde{\kappa}(z) = \mathrm{div}\frac{\nabla\tilde{u}}{|\nabla\tilde{u}|} = -\frac{\nabla\tilde{u}\cdot|\nabla\tilde{u}|^2}{2|\nabla\tilde{u}|^3} = -\frac{\mathrm{Re}(\tilde{U}''(\overline{\tilde{U}'})^2)}{2|U'|^3} = -\frac{1}{2}\mathrm{Re}(\tilde{U}''(\overline{\tilde{U}'})^2)$$

we have, in view of $|\tilde{\psi}'(z)| = 1 = |V'_{2a/r_0}(\tilde{\psi}(z))|$,

$$\begin{aligned}\tilde{\kappa} &= -\frac{1}{2}\mathrm{Re}\Big(\big(V''_{2a/r_0}\tilde{\psi}'^2 + V'_{2a/r_0}\tilde{\psi}''\big)\big(\overline{V'_{2a/r_0}\tilde{\psi}'}\big)^2\Big) = -\frac{1}{2}\mathrm{Re}\big(V''_{2a/r_0}(\overline{V'_{2a/r_0}})^2 + \tilde{\psi}''\overline{V'_{2a/r_0}(\tilde{\psi})'^2}\big)\\ &= \kappa_{2a/r_0}\circ\tilde{\psi} + O(\delta),\end{aligned}$$

which is the rescaled version of (34). □

We now have all the ingredients for Theorems 1.2 and Theorem 1.3.

Proof of Theorem 1.2. Fix $\delta > 0$ to be smaller than the flatness constant δ_0 (Remark 8.1). Let r_0, ϵ_0 be as in Theorem 9.1. Set $\delta_1 = \delta$, $M = 8\epsilon_0$ and apply Proposition 10.3 to find $s_1 = s_1(\delta, M)$. Finally, set $\epsilon_1 = \min\{s_1, \epsilon_0\}$. For any point $q \in F(u)$, let Z_q be the component of the zero phase to which q belongs. Define the set of points

$$\mathcal{C}_{\mathrm{prox}} = \{q \in F(u) \cap \overline{B_{1/2}} : \mathrm{dist}(q, \{u = 0\} \setminus Z_q) < \epsilon_1 r_0\}$$

in whose neighborhood we expect to see a hairpin structure. According to Theorem 9.1, for every $q \in \mathcal{C}_{\text{prox}}$ there exists a $z(q) \in B_{r_0/3}(q)$ such that $F(u) \cap B_{r_0/2}(z)$ is an approximate hairpin centered at $z = z(q)$ in the sense that:

- $F(u) \cap B_{r_0/2}(z)$ consists of two arcs F_L and F_R;
- if $s = \operatorname{dist}(F_L, F_R)$, we have for some rotation ρ and functions $f, g : \mathbb{R} \to \mathbb{R}$ with $f < g$,

$$\{u = 0\} \cap \big(B_{r_0/2}(z) \setminus B_{4s/\epsilon_0}(z)\big) = z + \rho\{4s/\epsilon_0 < |x| < r_0/2 : f(x_1) \le |x_2| \le g(x_1)\}$$
$$\text{where} \quad \|f\|_{L^\infty} + \|g\|_{L^\infty} \le \delta r, \quad \|f'\|_{L^\infty} + \|g'\|_{L^\infty} \le \delta.$$

At the same time, Proposition 10.3 says that inside $B_{8s\epsilon_0}(z)$,

$$|u(z + \tilde{\rho}x) - H_a(x)| \le \delta a.$$

for $a = s/(2+\pi)$ and some rotation $\tilde{\rho}$. Since the free boundary outside $B_{8s\epsilon_0}(z)$ has to match with the one inside, we may take $\tilde{\rho} = \rho$.

A standard covering argument yields a finite number of disks $\{B_{r_0/2}(z_j)\}_{j=1}^N$, where $N \le N_0 = O(r_0^{-2})$, which cover $\mathcal{C}_{\text{prox}}$ with the centers z_j constructed as above. For points $p \in F(u) \cap B_{1/2} \setminus \bigcup_{j=1}^N B_{r_0/2}(z_j)$, we know that

$$\operatorname{dist}(p, \{u = 0\} \setminus Z_p) \ge \epsilon_1 r_0,$$

so by Proposition 8.2, the curvature of $F(u)$ at p is at most $\kappa := \kappa_0(\delta)$.

Defining $r := 4r_0$, $\epsilon := \epsilon_0/4(2+\pi)$, we get the precise form of the statements in Theorem 1.2. □

Proof of Theorem 1.3. Fix $0 < \delta < 1/100$ small and let $r_0 = r(\delta)/2$ where r is as in Theorem 1.2. Running the same covering argument in the proof above, we have a collection of disks $\{B_{4r_0}(p_j)\}$ for each of which Theorem 10.7 gives: a unique saddle point z_j of u in $B_{4r_0}(p_j)$ and an injective holomorphic map

$$\psi_j : B_{4r_0}(p_j)^+(u) \to \Omega_{a_j} \quad \text{where} \quad a_j = u(z_j)$$

with all the enumerated properties in Theorem 10.7. Defining $\phi_j : B_{2r_0} \cap \Omega_{a_j} \to \mathbb{R}^2$ by $\phi_j := \psi_j^{-1}$ we obtain the precise form of the statements in Theorem 1.3. □

11. The minimal surface analogue.

In [Tra14] (see Theorems 9 and 10) Traizet discovered a remarkable correspondence between global solutions of (1) with $|\nabla u| < 1$ and complete embedded minimal bigraphs (minimal surfaces symmetric with respect to a plane with the two halves, "above" and "below" the plane, being graphical). The correspondence is expressed via the Weierstrass representation formula for *immersed* minimal surfaces. Recall, if $X : M \subseteq \mathbb{R}^3$ denotes the minimal immersion, the coordinate X_3 is a harmonic function on M and one can locally define a harmonic conjugate X_3^*, so that

$$dh = dX_3 + i dX_3^*$$

is a well-defined holomorphic differential on M (viewed as a Riemann surface), the so-called *height differential*. Furthermore, the stereographically projected Gauss map $g : M \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function on M. The pair (g, dh) is called the Weierstrass data of the minimal surface and the minimal immersion X is given, up to translation, by

$$X(p) = (X_1(p), X_2(p), X_3(p)) = \operatorname{Re} \int_{p_0}^{p} \left(\frac{1}{2}(g^{-1} - g)dh, \frac{i}{2}(g^{-1} + g)dh, dh\right) \tag{35}$$

where p_0 is a fixed point in M. Conversely, if M is a Riemann surface, and (g, dh) is a pair of a meromorphic function and a holomorphic 1-form on M, satisfying certain compatibility conditions ([Oss64]), then (35) defines a minimal immersion of M in $\mathbb{R}^3$.

Traizet's brilliant insight was to define

$$g = 2\frac{\partial u}{\partial z} \quad \text{and} \quad dh = 2\frac{\partial u}{\partial z} dz$$

in terms of a solution u of (1), and show that, under certain conditions, the Weierstrass data (g, dh) give rise to the upper half ($X_3 > 0$) of a minimal bigraph. Conversely, a solution u of (1) can be constructed using the Weierstrass data of a complete embedded minimal bigraph.

We have used Traizet's correspondence to state Corollary 1.4, the minimal surface version of Theorem 1.3. We can now turn to the proof.

Proof of Corollary 1.4. Following the argument of [Tra14, Theorem 10], we shall construct a solution of (1), corresponding to the minimal bigraph M. Let ζ be a complex coordinate on M, let g be the stereographically projected Gauss map and $dh = (2\partial X_3/\partial\zeta)\, d\zeta$ be the height differential. Note that $|g| = 1$ on $M \cap \{X_3 = 0\}$ as the normal points horizontally there and we may assume that the orientation of M is chosen so that the normal points down in M^+, i.e. $|g| < 1$ in M^+. Furthermore g has the same zeros and poles as dh (with same multiplicities), thus $g^{-1}dh$ defines a holomorphic non-vanishing one-form on M^+. Since M^+ is simply-connected,

$$\varphi(p) = \int_0^p g^{-1}dh \quad p \in M^+$$

defines a holomorphic function on M^+ (recall $0 \in M$). Claim that φ is injective. Define

$$\Xi := X_1 + iX_2$$

on M^+ and let $\hat{\Omega} = \Xi(M^+)$ be the projection of M^+ down to the horizontal plane $\{X_3 = 0\}$. Since M^+ is a graph, Ξ is a diffeomorphism from $\overline{M^+}$ to $\overline{\hat{\Omega}}$, so φ will be injective if and only if $\phi := \varphi \circ \Xi^{-1}$ is injective on $\hat{\Omega}$. Let a, b be arbitrary points of $\hat{\Omega}$ and let $[a, b] \subseteq \mathbb{C}$ denote the straight-line closed segment from a to b. Then for some $N \in \mathbb{N}$ we can write

$$[a,b] = \bigcup_{k=1}^{N} [z_{2k-1}, z_{2k}] \cup \bigcup_{k=1}^{N-1} [z_{2k}, z_{2k+1}],$$

where $z_1 = a$, $z_{2N} = b$, the interior of $[z_{2k-1}, z_{2k}]$ belongs to $\hat{\Omega}$, while z_{2k} and z_{2k+1} belong to the same connected component of $\partial\hat{\Omega}$. Claim that

$$\langle \overline{\phi}(z_{2k}) - \overline{\phi}(z_{2k-1}), \frac{b-a}{|b-a|} \rangle > |z_{2k} - z_{2k-1}|, \tag{36}$$

where $\langle w_1, w_2 \rangle := \mathrm{Re}(\overline{w_1} w_2)$ denotes the standard inner product on $\mathbb{C}$. Let $\alpha : [0,1] \to M^+$ be such that $\Xi \circ \alpha$ is the constant speed parameterization of $[z_{2k-1}, z_{2k}]$. For each fixed time $t \in (0,1)$, denote

$$v := \frac{1}{2}\overline{g^{-1}dh}(\alpha'(t)) \qquad w := -\frac{1}{2} g\,dh(\alpha'(t)),$$

we have $|v| > |w|$ because $|g| < 1$. Since $d\varphi(\alpha'(t)) = g^{-1}dh(\alpha'(t)) = 2\overline{v}$ and

$$z_{2k} - z_{2k-1} = d\Xi(\alpha'(t)) = (dX_1 + i dX_2)(\alpha'(t)) = \frac{1}{2}\overline{g^{-1}dh}(\alpha'(t)) - \frac{1}{2} g\,dh(\alpha'(t)) = v + w,$$

we have

$$\langle \overline{d\varphi}(\alpha'(t)), \frac{b-a}{|b-a|} \rangle = |z_{2k} - z_{2k-1}|^{-1} \langle 2v, v+w \rangle > |z_{2k} - z_{2k-1}|^{-1} |v+w|^2 = |z_{2k} - z_{2k-1}|$$

which leads to (36) once we integrate in t from 0 to 1. On the other hand,

$$\langle \overline{\phi}(z_{2k+1}) - \overline{\phi}(z_{2k}), \frac{b-a}{|b-a|} \rangle = |z_{2k+1} - z_{2k}|. \tag{37}$$

This is the case, because on the component β of $M \cap \{X_3 = 0\}$, to which $\Xi^{-1}(z_{2k+1})$ and $\Xi^{-1}(z_{2k})$ belong, we know $g^{-1} = \overline{g}$ and $\overline{dh} = -dh$, so that

$$\overline{d\varphi}(\beta') = \overline{g^{-1}dh}(\beta') = -g\,dh(\beta') = \frac{1}{2}\overline{g^{-1}dh}(\beta') - \frac{1}{2} g\,dh(\beta') = d\Xi(\beta')$$

and thus, $\overline{\phi}(z_{2k+1}) - \overline{\phi}(z_{2k}) = z_{2k+1} - z_{2k}$. Adding up (36) and (37) from $k = 1$ to N, we derive

$$\langle \phi(b) - \phi(a), (b-a)/|b-a| \rangle > |b-a|$$

from which the injectivity of ϕ follows.

We can now define the function

$$u = X_3 \circ \varphi^{-1}$$

on the domain $\Omega = \varphi(M^+)$, and we can easily verify that u is a positive, harmonic function in Ω that vanishes on $\partial\Omega \cap B_R$ where, for $z = \varphi(\zeta) \in F(u)$

$$|\nabla u|(z) = \left|2\frac{\partial X_3}{\partial \zeta}\frac{1}{\varphi'(\zeta)}\right| = |g(\zeta)| = 1.$$

Furthermore, the metric induced on Ω by the conformal immersion $X \circ \varphi^{-1}$ is given by the standard formula

$$ds = \frac{1}{2}(|g||dh| + |g|^{-1}|dh|) = \frac{1}{2}(|g|^2 + 1)|dz| = \lambda(z)|dz|$$

where $\frac{1}{2} \leq \lambda(z) \leq 1$, as $|g| \leq 1$ on M^+. So, if $\gamma^+ = \gamma \cap M^+$ denotes the piece of the shortest geodesic lying in M^+, it is mapped by φ to a curve $\tilde{\gamma} = \varphi(\gamma^+) \subseteq \Omega$ with Euclidean length $O(\mathcal{H}^1(\gamma^+)) = O(\epsilon)$ which connects the two pieces of $\partial\Omega$.

Fix $\delta < 1/1000$ a small positive numerical constant and let r_0, ϵ_1 be as in Theorem 10.7. Set $R_0 = 1/r_0$ and $\epsilon_0 = \epsilon_1$. Extend u by zero in $B_{R_0} \setminus \Omega$. Then u is a solution of (1) in B_{R_0}, satisfying (3), so Theorem 10.7 gives us an injective conformal map $\tilde{\psi} : B_4^+(u) \to \Omega_a$ for some appropriate $a = O(\epsilon)$, such that

$$\tilde{\psi}'(z) = 1 + O(\delta(|z| + a)), \qquad \tilde{\psi}''(z) = O(\delta) \qquad \text{for} \quad z \in B_4^+(u) \tag{38}$$

and $U := V_a \circ \tilde{\psi}$ is a holomorphic extension of u in $B_4^+(u)$ (recall V_a is the holomorphic extension of H_a given in Section 10). It's easy to see that $\tilde{\psi}$ gives rise to an injective conformal map from $M^+ \cap \mathcal{B}_2$ into the standard catenoid $\Sigma_\rho^+ := \Sigma_\rho \cap \{X_3 > 0\}$ (the counterpart to Ω_a in the Traizet correspondence), which then extends by symmetry to a conformal map ψ on all of $M \cap \mathcal{B}_2$. The metric on Ω_a induced by its immersion as Σ_ρ is

$$ds_{\text{cat}} = (1 + |V_a'|^2)|dz|$$

while the metric on $B_4^+(u)$ is

$$ds = (1 + |U'|^2)|dz|$$

and we check that the pull-back metric $\tilde{\psi}^*(ds_{\text{cat}})$ satisfies

$$\tilde{\psi}^*(ds_{\text{cat}}) = (1 + |U'|^2/|\tilde{\psi}'|^2)|\tilde{\psi}'||dz| = \big(1 + O(\delta(|z| + a))\big)ds.$$

Since $a \sim \rho$, the induced conformal map ψ is an isometry up to a factor of $\big(1 + O(\delta(|x| + \rho)\big)$and

$$\epsilon = \mathcal{H}^1(\gamma) = (1 + O(\delta))2\pi\rho \quad \Longrightarrow \quad |\epsilon - 2\pi\rho| = O(\delta\epsilon) < \epsilon/100.$$

Furthermore, the Gauss curvature of M is given by the standard formula for the curvature of a conformal metric $\lambda(z)|dz| = (1 + |U'|^2)|dz|$

$$K = -\frac{\Delta \log \lambda(z)}{\lambda^2(z)} = -\frac{4|U''|^2}{(1 + |U'|^2)^4}$$

Plugging in $U(z) = V_a(\tilde{\psi}(z))$ and applying the estimates (38), we get

$$\begin{aligned} K &= -\frac{4|V_a''(\tilde{\psi}')^2 + V_a'\tilde{\psi}''|^2}{(1 + |V_a'|^2|\tilde{\psi}'|^2)^4} = -\frac{4|V_a''|^2}{(1 + |V_a'|^2)^4}(1 + O(\delta(|z| + \rho)) + O\Big(\delta\frac{2|V_a''|}{(1 + |V_a'|^2)^2}\Big) + O(\delta^2) = \\ &= K_\rho + O\big(\delta(r + \rho)K_\rho\big) + O(\delta\sqrt{|K_\rho|}) + O(\delta^2) \end{aligned}$$

Noting that

$$\sqrt{|K_\rho(q)|} = O(\rho/r(q)^2)$$

and that $|r(\psi(p)) - r(p)| \sim \rho + \delta r(p)$ we obtain the desired estimate

$$K(p) = K_\rho(\psi(p)) + O\left(\delta\frac{\rho}{r + \rho}\sqrt{|K_\rho|}\right) + O\left(\delta\sqrt{|K_\rho|}\right) + O(\delta^2) = K_{\rho_0} + O\left(\delta + \sqrt{|K_\rho|}\right)\delta.$$

$\square$

References

[AC81] H. W. Alt and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.*, 325:105–144, 1981.

[ACF84] Hans Wilhelm Alt, Luis A. Caffarelli, and Avner Friedman. Variational problems with two phases and their free boundaries. *Trans. Amer. Math. Soc.*, 282(2):431–461, 1984.

[Caf87] Luis A. Caffarelli. A Harnack inequality approach to the regularity of free boundaries. I. Lipschitz free boundaries are $C^{1,\alpha}$. *Rev. Mat. Iberoamericana*, 3(2):139–162, 1987.

[Caf89] Luis A. Caffarelli. A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz. *Comm. Pure Appl. Math.*, 42(1):55–78, 1989.

[CJK04] Luis A. Caffarelli, David Jerison, and Carlos E. Kenig. Global energy minimizers for free boundary problems and full regularity in three dimensions. In *Noncompact problems at the intersection of geometry, analysis, and topology*, volume 350 of *Contemp. Math.*, pages 83–97. Amer. Math. Soc., Providence, RI, 2004.

[CM02] Tobias H Colding and William P Minicozzi. On the structure of embedded minimal annuli. *International Mathematics Research Notices*, 2002(29):1539–1552, 2002.

[CM04a] Tobias H. Colding and William P. Minicozzi, II. The space of embedded minimal surfaces of fixed genus in a 3-manifold. I. Estimates off the axis for disks. *Ann. of Math. (2)*, 160(1):27–68, 2004.

[CM04b] Tobias H. Colding and William P. Minicozzi, II. The space of embedded minimal surfaces of fixed genus in a 3-manifold. II. Multi-valued graphs in disks. *Ann. of Math. (2)*, 160(1):69–92, 2004.

[CM04c] Tobias H. Colding and William P. Minicozzi, II. The space of embedded minimal surfaces of fixed genus in a 3-manifold. III. Planar domains. *Ann. of Math. (2)*, 160(2):523–572, 2004.

[CM04d] Tobias H. Colding and William P. Minicozzi, II. The space of embedded minimal surfaces of fixed genus in a 3-manifold. IV. Locally simply connected. *Ann. of Math. (2)*, 160(2):573–615, 2004.

[CS05] Luis Caffarelli and Sandro Salsa. *A geometric approach to free boundary problems*, volume 68 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[DSJ11] Daniela De Silva and David Jerison. A gradient bound for free boundary graphs. *Comm. Pure Appl. Math.*, 64(4):538–555, 2011.

[HHP11] Laurent Hauswirth, Frédéric Hélein, and Frank Pacard. On an overdetermined elliptic problem. *Pacific J. Math.*, 250(2):319–334, 2011.

[JS] David Jerison and Ovidiu Savin. Some remarks on stability of cones for the one-phase free boundary problem. *preprint*. arXiv:1410.7463.

[KLT13] Dmitry Khavinson, Erik Lundberg, and Razvan Teodorescu. An overdetermined problem in potential theory. *Pacific J. Math.*, 265(1):85–111, 2013.

[KN77] D. Kinderlehrer and L. Nirenberg. Regularity in free boundary problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(2):373–391, 1977.

[Oss64] Robert Osserman. Global properties of minimal surfaces in e3 and en. *Annals of Mathematics*, pages 340–364, 1964.

[Tra14] Martin Traizet. Classification of the solutions to an overdetermined elliptic problem in the plane. *Geom. Funct. Anal.*, 24(2):690–720, 2014.

[Wei98] Georg S. Weiss. Partial regularity for weak solutions of an elliptic free boundary problem. *Comm. Partial Differential Equations*, 23(3-4):439–455, 1998.

David Jerison, Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

E-mail address: jerison@math.mit.edu

Nikola Kamburov, Department of Mathematics, The University of Arizona, 617 N Santa Rita Ave, P.O. Box 210089, Tucson AZ 85721, USA

E-mail address: kamburov@math.arizona.edu