

Singularities of the wave trace for the Friedlander model

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Abstract

In a recent preprint, [CGJ], we showed that for the Dirichlet Laplacian Δ on the unit disk, the wave trace $\text{Tr}(e^{it\sqrt{\Delta}})$, which has complicated singularities on $2\pi - \varepsilon < t < 2\pi$, is, on the interval $2\pi < t < 2\pi + \varepsilon$, the restriction to this interval of a C^∞ function on its closure. In this paper we prove the analogue of this somewhat counter-intuitive result for the Friedlander model. The proof for the Friedlander model is simpler and more transparent than in the case of the unit disk.

Introduction

Let Ω be a smooth bounded strictly convex region in the plane, and let $0 < \lambda_1 < \lambda_2 \leq \dots$ be the eigenvalues, with multiplicity, of the Dirichlet problem for the Laplacian on Ω :

$$\Delta u_j = \lambda_j u_j \quad \text{on } \Omega; \quad u_j = 0 \quad \text{on } \partial\Omega, \quad (\Delta = -(\partial_{x_1}^2 + \partial_{x_2}^2))$$

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In the spirit of [CdV0, Ch, DG, AM, GM]) we will define the *wave trace* of the Dirichlet Laplacian Δ to be the sum

$$Z(t) = \sum_{j=1}^{\infty} e^{it\sqrt{\lambda_j}}. \quad (1)$$

By a theorem of Andersson and Melrose [AM], $\operatorname{Re} Z(t)$ is a tempered distribution whose singular support in $t > 0$ is the closure of the set

$$\mathcal{L} = \bigcup_{k=2}^{\infty} \mathcal{L}_k \quad (2)$$

where \mathcal{L}_k is the set of critical values of the length function given by

$$|x_1 - x_2| + |x_2 - x_3| + \cdots + |x_k - x_1|, \quad (3)$$

in which $x_1 \dots x_k$ are distinct boundary points.

Each critical point of the length function (3) defines a polygonal trajectory consisting of the line segments joining x_i to x_{i+1} and x_k to x_1 . These are, by definition, the *closed geodesics* on Ω , and \mathcal{L} is the set of lengths of these geodesics, i.e. the *length spectrum* of Ω . Each set \mathcal{L}_k is closed, but there are limiting geodesics as $k \rightarrow \infty$. These limits are closed curves, called *gliding rays*, that lie entirely in $\partial\Omega$.

Near T in \mathcal{L} , the nature of the singularity of $Z(t)$ is fairly well understood in generic cases (see [GM]); but, at least at present, this is not so for T in the closure of the length spectrum. For example, if T is the perimeter of Ω , then there are trajectories forming a convex polygon with k sides whose perimeters tend to T as $k \rightarrow \infty$. In general, the limiting lengths T are integer multiples of the perimeter of Ω , and the set $(T - \varepsilon, T) \cap \mathcal{L}$ is infinite for every $\varepsilon > 0$. The cumulative effect of infinitely many singularities could be quite complicated and has not been analyzed completely even in simple model examples. However to the right of T one doesn't encounter this problem. $(T, T + \varepsilon) \cap \mathcal{L} = \emptyset$ for sufficiently small $\varepsilon > 0$, so that $\operatorname{Re} Z(t)$ is C^∞ for $t \in (T, T + \varepsilon)$. Therefore, several years ago the second author posed the question, what is the asymptotic behavior of $Z(t)$ as $t \rightarrow T$ from the right?

In one simple case, the case of the unit disk, we showed in [CGJ] that $Z(t)$ is C^∞ to the right, i.e., for some $\varepsilon > 0$, is C^∞ on $[T, T + \varepsilon)$. We will publish details of that result elsewhere; however in this article we will discuss another example for which we have been able to show that this assertion is

true, and, in fact, for which the verification is simpler and more transparent than in the case of the disk. This is the operator L introduced by F. G. Friedlander in [F], defined by

$$L = -\partial_x^2 - (1+x)\partial_y^2, \quad x \in [0, \infty), \quad y \in \mathbb{R}/2\pi\mathbb{Z} \quad (4)$$

on the manifold $M = [0, \infty) \times (\mathbb{R}/2\pi\mathbb{Z})$ with Dirichlet boundary conditions at $x = 0$. It represents the simplest approximation to the Laplace operator near the boundary of a convex domain.

The operator L has continuous spectrum on the space of functions that depend on x alone. On the orthogonal complement of this space, the spectrum is discrete, and the eigenfunctions of L are described in terms of the Airy function Ai as follows (see Appendix A (28)).

Proposition 1. *The Dirichlet eigenfunctions of L on*

$$\{\varphi \in L^2(M) : \int_{\mathbb{R}/2\pi\mathbb{Z}} \varphi(x, y) dy = 0\}$$

are the functions

$$\varphi_{m,n}(x, y) = \text{Ai}(n^{2/3}x - t_m)e^{iny}, \quad m = 1, 2, \dots, \quad n = \pm 1, \pm 2, \dots; \quad (5)$$

and their eigenvalues are

$$\lambda(m, n) = n^2 + n^{4/3}t_m, \quad (6)$$

in which $0 < t_1 < t_2 < \dots$ are the roots of $\text{Ai}(-t) = 0$.

We will define the trace of the wave operator associated to L by

$$Z_F(t) = \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{it\sqrt{\lambda(m,n)}}. \quad (7)$$

Our main result is as follows.

Theorem 1. *Let \mathcal{L}_F be the Friedlander length spectrum, as computed in Section 1.*

- a) *The limit points of \mathcal{L}_F are $\bar{\mathcal{L}}_F \setminus \mathcal{L}_F = 2\pi\mathbb{N}$.*
- b) *Z_F is C^∞ in the complement of $\bar{\mathcal{L}}_F \cup (-\bar{\mathcal{L}}_F) \cup \{0\}$*
- c) *For each $\ell \in \mathbb{N}$, there is $\varepsilon_\ell > 0$ such that Z_F extends to a C^∞ function on $[2\pi\ell, 2\pi\ell + \varepsilon_\ell)$.*

The main issue in the proof is to identify appropriate non-classical symbol properties for the function $\lambda(m, n)$. The Friedlander model is significantly easier to treat than other integrable models such as the disk because the asymptotics of the spectral function $\lambda(m, n)$ follow easily from well known asymptotics of the Airy function.

The paper is organized as follows. In Section 1, we find the length spectrum of L . In Section 2, we show how the Poisson summation formula is used to deduce smoothness of the trace when the spectral function is a classical symbol. Next, we show, in Section 3, that the Friedlander spectral function $\lambda(m, n)$ belongs to an appropriate non-classical symbol class of product type. In Section 4, we prove the smoothness properties of the Friedlander wave trace Z_F (Theorem 1) by dividing the (m, n) lattice into sectors and using Poisson summation methods adapted in each sector to the symbol properties of $\lambda(m, n)$. We conclude, in Section 5, by computing the action coordinates of the completely integrable system corresponding to the operator L and show that they capture the first order properties of eigenvalues and eigenfunctions of L in sectors where $\lambda(m, n)$ behaves like a classical symbol. This final computation reveals more details about which parts of the spectrum are linked to which aspects of the geodesic flow.

In Appendix A, we recall the properties of the Airy function and prove Proposition 1 characterizing the spectrum of L . In Appendix B we review the semiclassical analysis of joint eigenfunctions of commuting operators in a general integrable system, showing how to find the parametrization of the eigenfunctions by a discrete lattice and how to carry out the computations of Section 5 linking the classical and quantum systems in a more general context.

1 Closed geodesics

Our first task is to calculate the length spectrum \mathcal{L}_F for the Friedlander model. Define the Hamiltonian function H as the square root of the symbol of L ,

$$H(x, y; \xi, \eta) = \sqrt{\xi^2 + (x + 1)\eta^2}$$

The integral curves of the Hamiltonian vector field of H restricted to the hypersurface $H = 1$ are defined as solutions to

$$\dot{x} = \xi, \quad \dot{y} = (1 + x)\eta, \quad \dot{\xi} = -\eta^2/2, \quad \dot{\eta} = 0 \tag{8}$$

with the constraint

$$\xi^2 + (1 + x)\eta^2 = 1 \quad (9)$$

We begin by analyzing a single trajectory between boundary points.

Proposition 2. *Consider an integral curve of the Hamiltonian system (8)-(9) on $0 \leq t \leq T$, with*

$$x(0) = 0, \quad x(t) > 0 \text{ for } 0 < t < T, \quad x(T) = 0$$

If the initial velocity is $\xi(0) = \xi_0 > 0$, $\eta(0) = \eta_0$, then final velocity is given by

$$\dot{x}(T) = -\xi_0, \quad \dot{y}(T) = \eta_0,$$

and

$$T = 4\xi_0/\eta_0^2; \quad y(T) - y(0) = 4\xi_0/\eta_0 + (8/3)\xi_0^3/\eta_0^3$$

Proof. Solving the equations, we get

$$\eta(t) = \eta_0; \quad \xi(t) = \xi_0 - \eta_0^2 t/2, \quad x(t) = \xi_0 t - \eta_0^2 t^2/4$$

By translation invariance in y , we may as well assume $y(0) = 0$. Then

$$y(t) = \eta_0 t + \eta_0 \xi_0 t^2/2 - \eta_0^3 t^3/12$$

and

$$x(T) = 0 \implies \xi_0 T - \eta_0^2 T^2/4 = 0 \implies T = 4\xi_0/\eta_0^2$$

Substituting for T we obtain

$$\dot{x}(T) = \xi_0 - \eta_0^2 T/2 = -\xi_0,$$

and

$$y(T) - y(0) = y(T) = \eta_0 T + \eta_0 \xi_0 T^2/2 - \eta_0^3 T^3/12 = 4\xi_0/\eta_0 + (8/3)\xi_0^3/\eta_0^3$$

Finally, $x(T) = 0$ and $\eta(T) = \eta_0$ imply

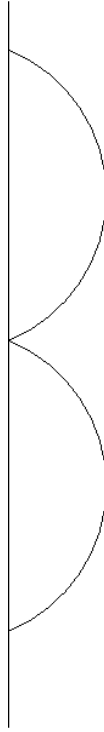
$$\dot{y}(T) = (1 + x(T))\eta(T) = \eta_0$$

which concludes the proof. □

We call a continuous curve $(x, y) : [a, b] \rightarrow M$ a *geodesic* if there are finitely many points $t_0 < t_1 < \dots < t_k$ for which the curve meets the boundary (i. e. $x(t_j) = 0$) and for $t_j < t < t_{j+1}$, the curve is the projection onto (x, y) of a solution to (8) and (9), and, finally, at each t_j the curve undergoes a reflection in the boundary:

$$\lim_{t \rightarrow t_j^+} \dot{x}(t) = - \lim_{t \rightarrow t_j^-} \dot{x}(t); \quad \lim_{t \rightarrow t_j^+} \dot{y}(t) = \lim_{t \rightarrow t_j^-} \dot{y}(t). \quad (10)$$

The Hamiltonian equations are translation invariant in t and y . Proposition 2 shows that a geodesic with starting velocity $(\dot{x}(t_j^+), \dot{y}(t_j^+)) = (\xi_0, \eta_0)$ has final velocity $(\dot{x}(t_{j+1}^-), \dot{y}(t_{j+1}^-)) = (-\xi_0, \eta_0)$. By (10), the reflected velocity $(\dot{x}(t_{j+1}^+), \dot{y}(t_{j+1}^+)) = (\xi_0, \eta_0)$, the same as the starting velocity. Consequently, each arc of a geodesic between encounters with the boundary is a translate of the others. This is analogous to equilateral polygonal trajectories in the disk. The figure below depicts a trajectory reflected at the boundary $x = 0$. Note also that it is tangent to the caustic, $x = \xi_0^2/\eta_0^2$, depicted by a dotted line.



Proposition 3. *A closed geodesic with $k = 1, 2, \dots$ reflections and winding number $\ell = \pm 1, \pm 2, \dots$ in the y variable, has length $L_{k,\ell} > 0$ determined by the equations*

$$L_{k,\ell} \frac{\eta_0^2}{\sqrt{1-\eta_0^2}} = 4k, \quad L_{k,\ell} \left(\frac{1}{3}\eta_0 + \frac{2}{3\eta_0} \right) = 2\pi\ell \quad (11)$$

Proof. Proposition 2 shows that the length of a closed geodesic with k reflections is kT , with T given as in the proposition. Hence, using $\xi_0^2 + \eta_0^2 = 1$,

$$L_{k,\ell} = kT = 4k\xi_0/\eta_0^2 \implies L_{k,\ell} \frac{\eta_0^2}{\sqrt{1-\eta_0^2}} = 4k.$$

Moreover, if the winding number is ℓ , then if $y(t)$ is as in Proposition 2 with $y(0) = 0$,

$$ky(T) = 2\pi\ell \implies k(4\xi_0/\eta_0 + (8/3)\xi_0^3/\eta_0^3) = 2\pi\ell$$

Substituting $L_{k,\ell} = 4k\xi_0/\eta_0^2$ into the last equation, it can be written

$$L_{k,\ell}(\eta_0 + (2/3)\xi_0^2/\eta_0) = 2\pi\ell \iff L_{k,\ell} \left(\frac{1}{3}\eta_0 + \frac{2}{3\eta_0} \right) = 2\pi\ell$$

□

The Friedlander length spectrum is defined by

$$\mathcal{L}_F = \{L_{k,\ell} : k \in \mathbb{N}, \ell \in \mathbb{Z} - \{0\}\}$$

where $L_{k,\ell}$ solve the equations of Proposition 3 for some η_0 , $-1 < \eta_0 < 1$. Note that if η_0 is replaced by $-\eta_0$, then ℓ is replaced by $-\ell$ and $L_{k,-\ell} = L_{k,\ell} > 0$. Thus we may confine our attention to $(k, \ell) \in \mathbb{N} \times \mathbb{N}$.

The implicit definition of $L_{k,\ell}$ can be restated as follows. Let

$$H : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$$

be the homogeneous function of degree 1 that is equal to 1 on the curve

$$u \rightarrow \left(\frac{\pi}{2} \frac{u^2}{\sqrt{1-u^2}}, \left(\frac{u}{3} + \frac{2}{3u} \right) \right) \quad 0 < u < 1. \quad (12)$$

Then $L_{k,\ell} = H(2\pi k, 2\pi\ell)$.

Proposition 4. *The Friedlander length spectrum \mathcal{L}_F can be written*

$$\mathcal{L}_F = H(2\pi\mathbb{N} \times 2\pi\mathbb{N}) \quad (13)$$

Furthermore,

- a) $\bar{\mathcal{L}}_F = \mathcal{L}_F \cup 2\pi\mathbb{N}$.
- b) For each $\ell \in \mathbb{N}$, there is $\varepsilon_\ell > 0$ such that $(2\pi\ell, 2\pi\ell + \varepsilon_\ell) \cap \mathcal{L}_F = \emptyset$.
- c) $\mathcal{L}_F \cap 2\pi\mathbb{N} = \emptyset$

Proof. Because the curve (12) is the graph of a decreasing function, $L_{k,\ell} < L_{k+1,\ell}$. Furthermore, $L_{1,\ell} \rightarrow \infty$ as $\ell \rightarrow \infty$. Therefore, every sequence of distinct values of $L_{k,\ell}$ tends to infinity unless ℓ remains bounded. It follows from this and the fact that the curve (12) is asymptotic to the horizontal line of height 1 that all the limit points are given by

$$\lim_{k \rightarrow \infty} L_{k,\ell} = 2\pi\ell \lim_{k \rightarrow \infty} H(k/\ell, 1) = 2\pi\ell$$

The sequence $L_{k,\ell}$ is increasing in k for fixed ℓ , so (a) and (b) are established. Finally, to confirm (c), note that if $2\pi m = L_{k,\ell}$ for some (k, ℓ) , then the equations (11) imply that π is algebraic, a contradiction. \square

Proposition 4 (a) and (c) imply Theorem 1 (a).

2 Poisson summation formula with classical symbols

The symbol class $S^m(\mathbb{R}^n)$ is the set of functions $a \in C^\infty(\mathbb{R}^n)$ satisfying

$$|\partial_\omega^\alpha a(\omega)| \leq C_\alpha (1 + |\omega|)^{m - \alpha_1 - \alpha_2 - \dots - \alpha_n}, \quad \omega \in \mathbb{R}^n \quad (14)$$

for all multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and some constants C_α . $S_{cl}^m(\mathbb{R}^n)$ consists of the symbols in $S^m(\mathbb{R}^n)$ with an asymptotic expansion, that is, there are homogeneous functions a_j of degree $m - j$, smooth except at the origin, and a cut-off function $\chi_N \in C^\infty(\mathbb{R}^n)$ such that $\chi_N = 1$ on $|\omega| \geq C_N$, and

$$\left(a - \sum_{j=0}^N a_j \right) \chi_N \in S^{m-N-1}(\mathbb{R}^2) \quad (15)$$

The function a_0 is the principal symbol of a . Functions defined on a cone $\Gamma \subset \mathbb{R}^n$ belong to the symbol class $S^m(\Gamma)$ ($S_{cl}^m(\Gamma)$) if they are restrictions of symbols in the class $S^m(\mathbb{R}^n)$ ($S_{cl}^m(\mathbb{R}^n)$).

Let F be a real-valued symbol of first order, $F \in S_{cl}^1(\mathbb{R}^2)$ with positive principal symbol. Let \mathbb{T}^2 be the 2-torus, $\mathbb{R}^2/2\pi\mathbb{Z}^2$, and let $P : C^\infty(\mathbb{T}^2) \rightarrow C^\infty(\mathbb{T}^2)$ be the constant coefficient self-adjoint first order elliptic pseudodifferential operator given by

$$P(e^{2\pi i(mx+ny)}) = F(m, n)e^{2\pi i(mx+ny)}$$

The eigenvalues of P are $F(m, n)$, $(m, n) \in \mathbb{Z}^2$, and hence its wave trace is

$$Z(t) = \sum_{(m,n) \in \mathbb{Z}^2} e^{itF(m,n)}. \quad (16)$$

Since the summands are smooth functions of t , the singularities of $Z(t)$ depend only on the asymptotic behavior of $F(m, n)$ as $(m, n) \rightarrow \infty$. For simplicity we will assume that $F = F_1$ on $\xi^2 + \eta^2 \geq 1/2$, where F_1 is homogeneous of degree 1 and positive. (With small modifications the proof we will give applies in general.)

To identify the length spectrum, consider the Hamiltonian flow on the cotangent bundle of \mathbb{T}^2 associated with the symbol, F_1 , defined by the ordinary differential equations

$$\dot{x} = \partial_\xi F_1, \quad \dot{y} = \partial_\eta F_1, \quad \dot{\xi} = \dot{\eta} = 0, \quad (17)$$

Its periodic trajectories are curves of the form, $(x, y) + t\nabla F_1(\xi, \eta)$, $0 \leq t \leq T$, satisfying

$$T\nabla F_1(\xi, \eta) = 2\pi(p, q) \quad (18)$$

for some $T > 0$ and some $(p, q) \in \mathbb{Z}^2$. Let $T_{p,q}$ be the set of T for which (18) has a solution for some $(\xi, \eta) \neq (0, 0)$. (Note that ∇F_1 is homogeneous of degree 0, so that we may as well assume $\xi^2 + \eta^2 = 1$.) The length spectrum is given by

$$\mathcal{L} = \bigcup_{(p,q) \in \mathbb{Z}^2} T_{p,q}$$

Note that because ∇F_1 is nonzero, bounded, and continuous on $\xi^2 + \eta^2 = 1$, the length spectrum is closed and bounded away from 0. Our goal is to prove that the function $Z(t)$ in (16) is C^∞ in $t > 0$ on the complement of the length spectrum.

To do so we will apply the Poisson summation formula to the function, $e^{itF(\xi,\eta)}$. The Fourier transform of this function is

$$\int_{\mathbb{R}^2} e^{i(tF(\xi,\eta)-2\pi(x\xi+y\eta))} d\xi d\eta$$

so by the Poisson summation formula,

$$Z(t) = \sum_{(p,q) \in \mathbb{Z}^2} Z_{p,q}(t) \quad (19)$$

where

$$Z_{p,q}(t) = \int_{\mathbb{R}^2} e^{i(tF(\xi,\eta)-2\pi(p\xi+q\eta))} d\xi d\eta. \quad (20)$$

We will first show that on the complement of the length spectrum these individual summands are C^∞ . If $t_0 > 0$ and $t_0 \notin \mathcal{L}$ then there exists an interval, $t_0 - \epsilon \leq t \leq t_0 + \epsilon$, such that for all $|(\xi, \eta)| \geq 1$, either $t\partial_\xi F(\xi, \eta) - 2\pi p$ or $t\partial_\eta F - 2\pi q$ is non-zero. Hence for t on this interval $Z_{p,q}(t)$ can be written as a sum

$$\sum_{r=1}^3 \int_{\mathbb{R}^2} e^{i(tF-2\pi(p\xi+q\eta))} \rho_r(\xi, \eta) d\xi d\eta \quad (21)$$

where ρ_1 and ρ_2 are smooth homogeneous functions of degree zero on $|(\xi, \eta)| \geq 1$ and the first of the functions above is non-zero on $\text{supp } \rho_1$ and the second non-zero on $\text{supp } \rho_2$. Finally ρ_3 is supported in a compact neighborhood of the origin. The third integrand is compactly supported, and the corresponding integral converges along with all derivatives with respect to t , regardless of the range of t . Let

$$a_0(\xi, \eta, p, t) = i(t\partial_\xi F - 2\pi p)^{-1}. \quad (22)$$

Then

$$e^{i(tF-2\pi(p\xi+q\eta))} \rho_1 = a_0 \partial_\xi (e^{i(tF-2\pi(p\xi+q\eta))}) \rho_1$$

hence by integration by parts the first summand of (21) can be written as

$$\int_{\mathbb{R}^2} e^{i(tF-2\pi(p\xi+q\eta))} a_1 d\xi d\eta$$

or inductively as

$$\int_{\mathbb{R}^2} e^{i(tF-2\pi(p\xi+q\eta))} a_N d\xi d\eta$$

where

$$a_1 = -\partial_\xi(a_0\rho_1); \quad a_N(\xi, \eta, p, t) = -\partial_\xi(a_0a_{N-1}). \quad (23)$$

Since a_0 is a homogeneous function of degree zero in (ξ, η) outside a small neighborhood of the origin, it follows by induction from (23) that a_N is a homogeneous function of degree $-N$, and hence

$$|a_N(\xi, \eta, p, t)| \leq C_N(1 + |\xi|^2 + |\eta|^2)^{-N/2} \quad (24)$$

uniformly on the interval $-\epsilon + t_0 \leq t \leq \epsilon + t_0$. Differentiation k times with respect to t introduces the factor F^k in the integrand, which is dominated by $|(\xi, \eta)|^k$; hence the integral converges. In all, the first of the summand in (21) is a C^∞ function of t in the range specified. The proof for the term ρ_2 is similar, replacing a_0 by $b_0 = i(t\partial_\eta F - 2\pi q)^{-1}$, a_1 by $b_1 = -\partial_\eta(b_0\rho_2)$ and a_N by $b_N = -\partial_\eta(a_0b_{N-1})$ one concludes by the same argument that the second of these summands is C^∞ using integration by parts in η .

For $|p|$ large, $t\partial F/\partial\xi - 2\pi p$ is non-vanishing for all (ξ, η) and all t in $t_0 - \epsilon \leq t \leq t_0 + \epsilon$, and integration by parts with respect to ξ yields

$$Z_{p,q}(t) = \int_{\mathbb{R}^2} e^{i(tF - 2\pi(p\xi + q\eta))} \tilde{a}_N d\xi d\eta, \quad N = 1, 2, \dots$$

with $\tilde{a}_0 = 1$ and

$$\tilde{a}_N = -\partial_\xi(a_0\tilde{a}_{N-1})$$

The estimate (24) is replaced by

$$|\tilde{a}_N(\xi, \eta, p, t)| \leq C_N(1 + |\xi|^2 + |\eta|^2)^{-N/2}|p|^{-(N+1)}, \quad N \geq 1 \quad (25)$$

Similarly, for $|q|$ large, integration by parts with respect to η yields an analogous formula with better decay as $q \rightarrow \infty$. Hence if we break the sum $\sum Z_{p,q}$ into subsums, $\sum Z_{p,q}$, $|p| < |q|$, and $\sum Z_{p,q}$, $|p| \geq |q|$, and make use of these estimates we conclude that not only are the individual summands of (19) C^∞ but that the sum itself is C^∞ on the complement of the length spectrum \mathcal{L} .

3 Non-classical symbols

Let Γ be a cone (sector with vertex at the origin) in \mathbb{R}^2 .

Definition 1. A smooth function $a : \Gamma \rightarrow \mathbb{C}$ belongs to $\Sigma^{\alpha,\beta}(\Gamma)$ if there exists constants $C_{j,k}$ so that for all $(\xi, \eta) \in \Gamma$, the following upper bounds hold

$$\forall j, k \geq 0, |\partial_\xi^j \partial_\eta^k a(\xi, \eta)| \leq C_{j,k} (1 + |\xi|)^{\alpha-j} (1 + |\eta|)^{\beta-k}.$$

A typical example is $b(\xi)c(\eta) \in \Sigma^{\alpha,\beta}(\mathbb{R}^2)$ if $b \in S^\alpha(\mathbb{R})$ and $c \in S^\beta(\mathbb{R})$.

Proposition 5. The following properties hold.

- a) $S^0(\Gamma) \subset \Sigma^{0,0}(\Gamma)$.
- b) If $a \in \Sigma^{\alpha,\beta}(\Gamma)$ and $b \in \Sigma^{\alpha',\beta'}(\Gamma)$, the product ab belongs to $\Sigma^{\alpha+\alpha',\beta+\beta'}(\Gamma)$.
- c) If $a \in \Sigma^{\alpha,\beta}(\Gamma)$ and $a \geq C(1+|\xi|)^\alpha(1+|\eta|)^\beta$ with $C > 0$ in Γ (a is said to be elliptic positive), then $a^{-1} \in \Sigma^{-\alpha,-\beta}(\Gamma)$. Moreover, $(a+c)^{-1} \in \Sigma^{-\alpha,-\beta}(\Gamma)$ uniformly for all constants $c \geq 0$.

To establish the symbol properties of the phase $\sqrt{\lambda(m,n)}$ in the formula (7) for the Friedlander wave trace Z_F , we will need a well known asymptotic formula for the roots of the Airy function. (A proof is given in Appendix A; see also [AS] page 450.)

Proposition 6. There is a symbol $\tau \in S_{cl}^{2/3}(\mathbb{R}_+)$ such that $\tau(\xi) > 0$ for all $\xi > 1/2$,

$$t_m = \tau(m), \quad m = 1, 2, \dots$$

are the roots of $\text{Ai}(-t) = 0$, and the principal symbol of $\tau(\xi)$ is $(3\pi\xi/2)^{2/3}$ ($\xi \rightarrow \infty$).

Denote

$$F(\xi, \eta) = (\eta^2 + \eta^{4/3}\tau(\xi))^{1/2}$$

Then $\sqrt{\lambda(m,n)} = F(m, |n|)$ and the sum (7) can be written

$$Z_F(t) = 2 \sum_{(m,n) \in \mathbb{N} \times \mathbb{N}} e^{itF(m,n)}$$

Let $0 < \kappa_1 < \kappa_2 < \infty$. The first quadrant is covered by the union of three overlapping sectors

$$\begin{aligned} \Gamma_1 &= \{(\xi, \eta) : |\xi| < \kappa_1 \eta\}; \\ \Gamma_2 &= \{(\xi, \eta) : (\kappa_1/2)\eta < \xi < 2\kappa_2 \eta\} \\ \Gamma_3 &= \{(\xi, \eta) : \kappa_2 |\eta| < \xi\} \end{aligned}$$

One can deduce easily from Proposition 6 the following.

Proposition 7. *Let $G(\xi, \eta) = F(\xi, \eta) - \eta$. Then*
a) $G \in \Sigma^{2/3, 1/3}(\Gamma_1)$, and $\partial_\eta G$ is positive, elliptic in $\Sigma^{2/3, -2/3}(\Gamma_1)$.
b) $F \in S_{cl}^1(\Gamma_2)$ with principal symbol F_0 given by

$$F_0(\xi, \eta) = \sqrt{\eta^2 + (3\pi\xi/2)^{2/3}\eta^{4/3}}$$

c) $F \in \Sigma^{1/3, 2/3}(\Gamma_3)$, and $\partial_\xi F$ is positive elliptic in $\Sigma^{-2/3, 2/3}(\Gamma_3)$.

4 Poisson summation formula with non-classical symbols

We are now ready to finish the proof of Theorem 1. First, we divide the sum representing Z_F into three regions corresponding to different behaviors of the symbol $F(\xi, \eta)$. Let $\chi_j \in S^0(\mathbb{R}^2)$ be supported in Γ_j and such that

$$\sum_{j=1}^3 \chi_j(\xi, \eta) = 1 \quad \text{for all } \xi > 0, \eta > 0, \xi^2 + \eta^2 \geq 1/4$$

Let $\psi \in C^\infty(\mathbb{R})$ satisfy $\psi(\xi) = 1$ for all $\xi \geq 1$ and $\psi(\xi) = 0$ for all $\xi \leq 1/2$. Then

$$Z_F(t) = 2 \sum_{j=1}^3 I_j(t), \quad I_j(t) = \sum_{(m,n) \in \mathbb{Z}^2} e^{itF(m,n)} \psi(m) \psi(n) \chi_j(m, n)$$

The method of Section 2 applies in the sector Γ_2 . The times t at which singularities can occur are the stationary points $T_{p,q}$ of the phases of the representation using the Poisson summation formula, namely, solutions to

$$T_{p,q} \nabla F_0(\xi, \eta) = 2\pi(p, q) \tag{26}$$

Furthermore, on $F_0(\xi, \eta) = 1$,

$$\nabla F_0(\xi, \eta) = \left(\frac{\pi}{2} \frac{\eta^2}{\sqrt{1-\eta^2}}, \frac{1}{3}\eta + \frac{2}{3}\eta^{-1} \right)$$

Thus if $(k, \ell) \in \mathbb{N} \times \mathbb{N}$, then $T_{k,\ell} = L_{k,\ell}$, exactly the points of the length spectrum \mathcal{L}_F . The set of $T_{p,q}$ is

$$-\mathcal{L}_F \cup \{0\} \cup \mathcal{L}_F$$

(Evidently, $T_{0,0} = 0$ is a stationary point; there are no roots $T_{0,q}$, $q \neq 0$ and no roots $T_{p,0}$, $p \neq 0$ of (26). The rest of the range $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ is covered by the symmetries $T_{k,-\ell} = T_{k,\ell}$ and $T_{-k,-\ell} = -T_{k,\ell}$.)

The restriction to Γ_2 gives a further limitation on the set of stationary points. Define a smaller collection of stationary points by

$$\mathcal{L}_F(M) = \{L_{k,\ell} : (k, \ell) \in \mathbb{N}^2, \quad 1 \leq k \leq M\ell\}$$

$\mathcal{L}_F(M)$ is a closed, discrete set in \mathbb{R} . For M sufficiently large depending on κ_1 , all the stationary values that occur in the the range $\xi \geq \kappa_1|\eta|/2$ are in $\mathcal{L}_F(M)$. Because $\mathcal{L}_F(M)$ has no limit points, and the symbol of the phase has classical behavior in Γ_2 , the method of Section 2 applies and yields the following.

Proposition 8. *For M sufficiently large depending on κ_1 , I_2 is smooth in the complement of*

$$\mathcal{L}_F(M) \cup (-\mathcal{L}_F(M)) \cup \{0\}$$

In particular, for any $\ell \in \mathbb{N}$ and any $\kappa_1 > 0$ there is $\varepsilon > 0$ such that

$$I_2 \in C^\infty((2\pi\ell - \varepsilon, 2\pi\ell + \varepsilon))$$

All the remaining singularities in $\bar{\mathcal{L}}_F - \mathcal{L}_F(M)$ are accounted for in Γ_1 , associated with I_1 . The main issue is the limit points at $t = 2\pi\ell$. Our theorem in Γ_1 is stated in sufficient generality to apply as well to the case of the disk.

Theorem 2. *Let $\psi(\xi)$ be a classical symbol of degree 0 in one variable supported in $\xi \geq 0$ and let $\chi(\xi, \eta)$ be a classical symbol of degree 0 supported in a cone Γ_1 given by $|\xi| \leq \kappa_1\eta$. Suppose that the distribution Z is defined by*

$$Z(t) = \sum_{(m,n) \in \mathbb{Z}^2} \psi(m)\chi(m, n)e^{it(n+G(m,n))} ,$$

where G belongs to $\Sigma^{2/3, 1/3}$, with $\partial_\eta G > 0$ is positive elliptic in $\Sigma^{2/3, -2/3}$ on the support of $\psi(m)\chi(m, n)$. For any $T > 0$ there exist $\varepsilon > 0$ and $\kappa_1 > 0$ sufficiently small depending on T and the constant in the symbol upper bound on $\partial_\eta G$ such that

- a) if $T = 2\pi\ell$, $\ell \in \mathbb{N}$, then Z is smooth on $[T, T + \varepsilon)$;*
- b) if $T \notin 2\pi\mathbb{N}$, then Z is smooth on $(T - \varepsilon, T + \varepsilon)$.*

Proof. In the proof of part (a) we will assume for simplicity that $\ell = 1$. The other cases are similar. Let $F(\xi, \eta) = \eta + G(\xi, \eta)$. The Poisson summation formula says

$$Z(t) = \sum_{(p,q) \in \mathbb{Z}^2} Z_{p,q}(t)$$

with

$$Z_{p,q}(t) = \int_{\mathbb{R}^2} \psi(\xi) \chi(\xi, \eta) e^{i(tF(\xi, \eta) - 2\pi(p\xi + q\eta))} d\xi d\eta ,$$

Denote $Q = i(tF(\xi, \eta) - 2\pi(p\xi + q\eta))$ and define two differential operators in order to integrate by parts as follows.

$$Af = e^{itF(\xi, \eta)} \frac{i}{2\pi} \partial_\xi (e^{-itF(\xi, \eta)} f) = \frac{i}{2\pi} \partial_\xi f + \frac{t}{2\pi} (\partial_\xi F) f .$$

$$Bf = \frac{1}{i(t\partial_\eta F(\xi, \eta) - 2\pi q)} \partial_\eta f$$

Then $\frac{1}{p} A(e^Q) = B(e^Q) = e^Q$.

Let us start with the most singular case $q = 1$. Because $G \in \Sigma^{2/3, 1/3}$, we have

$$\partial_\xi F = \partial_\xi G \in \Sigma^{-1/3, 1/3}, \quad \partial_\eta G \in \Sigma^{2/3, -2/3} .$$

The main point is that by Proposition 5,

$$\frac{\psi(\xi) \chi(\xi, \eta)}{t\partial_\eta F - 2\pi} = \frac{\psi(\xi) \chi(\xi, \eta)}{t - 2\pi + \partial_\eta G} \in \Sigma^{-2/3, 2/3}$$

uniformly in $t > 2\pi$. Let A' and B' denote the adjoint operators to A and B . Then Proposition 5 also implies that

$$A'(\Sigma^{\alpha, \beta}) \subset \Sigma^{\alpha-1/3, \beta+1/3}; \quad B'(\Sigma^{\alpha, \beta}) \subset \partial_\eta \Sigma^{\alpha-2/3, \beta+2/3} \subset \Sigma^{\alpha-2/3, \beta-1/3}$$

with constants in the estimates independent of p .

$$Z_{p,1}(t) = p^{-M} \int e^Q (A')^M (B')^N (\psi(\xi) \chi(\xi, \eta)) d\xi d\eta .$$

Since $\psi(\xi) \chi(\xi, \eta) \in \Sigma^{0,0}$, the symbol $p^{-M} (A')^M (B')^N (\psi(\xi) \chi(\xi, \eta))$ belongs to $\Sigma^{-(M+2N)/3, (M-N)/3}$ with constants $O(|p|^{-M})$. Note that symbols in $\Sigma^{\alpha, \beta}$ supported in the cone $0 \leq \xi \leq \eta$ are integrable if $\max(\alpha, 0) + \beta < -1$.

Therefore it suffices to take $N \geq M + 5$ to get a convergent integral. In case $p = 0$, take $M = 0$ and N large. Otherwise, take $M \geq 2$ so that the sum over p converges. So far we have shown that that sum over $(p, 1)$ is bounded. To prove smoothness in t , differentiate k times. This yields an extra factor $F^k \in \Sigma^{2k/3, k}$ in the integrand. Therefore,

$$\sum_{p \in \mathbb{Z}} (d/dt)^k Z_{p,1}(t)$$

is represented by sum of integrals of symbols $\Sigma^{2k/3 - M/3 - 2N/3, k + M/3 - N/3}$ with constants $O(|p|^{-M})$. The integrals converge for N large, for instance $N \geq M + 3k + 10$. Again, take $M = 0$ when $p = 0$ and $M \geq 2$ when $p \neq 0$.

Next, consider $q \neq 1$. The same estimates as before hold for A' . Recall that we may assume that χ is zero on a fixed compact subset. Provided t is close enough to 2π , and κ_1 sufficiently small, and $|\xi| + |\eta|$ sufficiently large on the support of χ , we have

$$|\partial_\eta G| \leq C((1 + |\xi|)/(1 + |\eta|))^{2/3} \leq C\kappa_1 \ll 1$$

and hence $|t\partial_\eta F - 2\pi q| \sim |q - 1|$ and

$$\psi(\xi)\chi(\xi, \eta)/(t\partial_\eta F - 2\pi q) \in \Sigma^{0,0}, \quad \text{with constants } O(1/|q - 1|)$$

Hence,

$$B' : \Sigma^{\alpha, \beta} \rightarrow \Sigma^{\alpha, \beta - 1}$$

with constants $O(1/|q - 1|)$. So the symbol $p^{-M}(A')^M(B')^N(\psi(\xi)\chi(\xi, \eta))$ belongs to $\Sigma^{-M/3, -N}$ with constants $O(|p|^{-M}|q - 1|^{-N})$. The integral converges provided $N \geq 2$. When $p = 0$, take $M = 0$; when $p \neq 0$ take $M \geq 2$. Then the sum over (p, q) , with $q \neq 1$ also converges. Lastly, if we apply $(d/dt)^k$, we obtain integrands that belong to $\Sigma^{2k/3 - M/3, k - N}$ with constants $O(|p|^{-M}|q - 1|^{-N})$, so the symbols are integrable if $N \geq 5k/3 + 2$. To sum the series, for $p \neq 0, q \neq 1$, let $M \geq 2$. For $p = 0, q \neq 1$, use $M = 0$. This concludes the proof of part (a) of Theorem 2.

For part (b), once again for any $\delta > 0$ we can choose $\kappa_1 > 0$ sufficiently small that on the support of χ (where $|\xi| + |\eta|$ is sufficiently large)

$$|\partial_\eta G| \leq C((1 + |\xi|)/(1 + |\eta|))^{2/3} \leq C\kappa_1^{2/3} < \delta$$

Let q_0 be the integer such that $|T - 2\pi q_0|$ is smallest, and choose both $\delta > 0$ and $\varepsilon > 0$ less than $|T - 2\pi q_0|/4$. Then for $t \in (T - \varepsilon, T + \varepsilon)$, $|t\partial_\eta F - 2\pi q_0| \geq |T - 2\pi q_0|/4$ on the support of χ . Hence,

$$\psi(\xi)\chi(\xi, \eta)/(t\partial_\eta F - 2\pi q) \in \Sigma^{0,0}, \quad \text{with constants } O(1/(1 + |q - q_0|))$$

and the rest of the proof proceeds as in the case $q \neq 1$ above. This concludes the proof of Theorem 2. \square

The next proposition will take care the integral I_3 .

Proposition 9. *Let $\psi \in C^\infty(\mathbb{R})$ satisfy $\psi(\eta) = 1$ for η sufficiently large and $\psi(\eta) = 0$ for $\eta \leq 0$. Suppose that where F belongs to $\Sigma^{1/3, 2/3}$ and $\partial_\xi F$ is positive elliptic in $\Sigma^{-2/3, 2/3}$ on the cone $0 < \eta < \xi$. For each $T < \infty$ there exists κ_2 such that if $\chi(\xi, \eta)$ is a classical symbol of degree 0 supported in a cone Γ given by $\xi > \kappa_2|\eta|$, then the distribution*

$$Z(t) = \sum_{(m,n) \in \mathbb{Z}^2} \psi(n)\chi(m, n)e^{itF(m,n)},$$

satisfies $Z \in C^\infty((0, T))$.

Proof. As usual, the Poisson summation formula implies

$$Z(t) = \sum_{(p,q) \in \mathbb{Z}^2} Z_{p,q}(t)$$

with

$$Z_{p,q}(t) = \int_{\mathbb{R}^2} \psi(\eta)\chi(\xi, \eta)e^{i(tF(\xi, \eta) - 2\pi(p\xi + q\eta))} d\xi d\eta,$$

Denote $Q = i(tF(\xi, \eta) - 2\pi(p\xi + q\eta))$.

Case 1. $p = 0$. Then $Q = i(tF - 2\pi q\eta)$ and

$$\begin{aligned} Z_{0,q}(t) &= \int \int e^Q \psi(\eta)\chi(\xi, \eta) d\xi d\eta = - \int \int e^Q \partial_\xi(\chi/\partial_\xi Q)\psi(\eta) d\xi d\eta \\ &= \dots = \int \int e^Q a_M(\xi, \eta)\psi(\eta) d\xi d\eta \end{aligned}$$

with $a_0 = \chi$, $a_{M+1} = -\partial_\xi(a_M/\partial_\xi Q)$. Since $\partial_\xi F$ is positive elliptic in $\Sigma^{-2/3, 2/3}(\Gamma)$,

$$1/\partial_\xi Q = 1/it\partial_\xi F$$

belongs to $\Sigma^{2/3, -2/3}$ with support on Γ and bounds depending on $1/t$. By induction, $a_M \in \Sigma^{-M/3, -2M/3}$ with support in Γ_3 . Thus $Z_{0,q}$ is represented by a convergent integral. Moreover, if $q = 0$, we may differentiate k times with respect to t to obtain an integrand dominated by $|a_M F^k|$ which is integrable if $k < M/3 - 2$. For $q \neq 0$, use

$$e^{2\pi i q \eta} = \frac{1}{2\pi i q} \partial_\eta e^{2\pi i q \eta}$$

to integrate by parts N times to obtain

$$(d/dt)^j Z_{0,q}(t) = \frac{\pm 1}{(2\pi i q)^N} \int \int e^Q \partial_\eta^N [(iF)^j e^{itF} \psi(\eta) a_M(\xi, \eta)] d\xi d\eta$$

The integrand is majorized by $(1 + |\xi|)^{j+(N-M)/3}$ which is convergent in Γ for sufficiently large M depending on j and N . (Note that because χ is supported in $\xi < |\eta|$, $\psi'(\eta)\chi(\xi, \eta)$ has compact support, and every term in which a derivative falls on ψ is convergent.) The factor $1/q^N$ makes the sum over q convergent as well.

Case 2. $p \neq 0$. We use the same integration by parts as in Case 1. Note that the set of $(m, n) \in \mathbb{Z}^2$ where $\psi(n)\chi(m, n) > 0$ and $m < \xi_0$ is finite. Any finite sum of exponentials is smooth in t , so we may assume without loss of generality that $\chi(\xi, \eta)$ is supported on $\xi > \xi_0$ for some large ξ_0 . It follows that

$$\partial_\xi F(\xi, \eta) \leq C\xi^{-2/3}\eta^{2/3}$$

on the support of $\psi(\eta)\chi(\xi, \eta)$. Furthermore, if κ_2 is sufficiently large, depending on T ,

$$0 \leq t\partial_\xi F \leq CT\kappa_2^{-2/3} \leq \pi, \quad \text{for all } 0 \leq t \leq T$$

Thus

$$\chi/\partial_\xi Q = 1/i(t\partial_\xi F - 2\pi p) \in \Sigma^{0,0}(\Gamma)$$

with bounds $O(1/|p|)$. By induction,

$$a_M \in \Sigma^{-M,0}$$

with bounds $O(1/|p|^M)$. The rest of the proof proceeds as in Case 1. If $q = 0$, then no more integrations by parts are needed. If $q \neq 0$, then one uses integration by parts in η as in Case 1 to introduce a factor $|q|^{-N}$ so as to be able to sum in both p and q . \square

Finally, we assemble these ingredients to prove Theorem 1. Part (a) was already proved at the end of Section 1. For part(b), choose $T \notin \bar{\mathcal{L}}_F \cup (-\bar{\mathcal{L}}_F) \cup \{0\}$, and choose ε so that

$$[T - \varepsilon, T + \varepsilon] \cap (\bar{\mathcal{L}}_F \cup (-\bar{\mathcal{L}}_F) \cup \{0\}) = \emptyset$$

Choose κ_1 sufficiently small depending on the symbol bounds and the distance from $[T - \varepsilon, T + \varepsilon]$ to $2\pi\mathbb{Z}$ and κ_2 sufficiently large depending on the size of T , we may apply Theorem 2 (b), Proposition 8 and Proposition 9, respectively, to I_1 , I_2 and I_3 to obtain smoothness in $(T - \varepsilon, T + \varepsilon)$. Note that Proposition 7 implies that the corresponding phases satisfy the appropriate symbol estimates. For part (c), apply Theorem 2 (a) to I_1 , to obtain smoothness in $[2\pi\ell, 2\pi\ell + \varepsilon)$. By Propositions 8 and 9 I_2 and I_3 are smooth near $2\pi\ell$ on both sides. This concludes the proof of Theorem 1.

5 Action variables

The classical mechanical system associated to the Friedlander model has two commuting conserved quantities,

$$H(x, y, \xi, \eta) = \sqrt{\xi^2 + (1+x)\eta^2}; \quad J(x, y, \xi, \eta) = \eta.$$

In this section we will show how the action variables associated to this system determine the first order asymptotics of the eigenvalues and eigenfunctions of the Friedlander operator L . (See Appendix B for the general framework of this calculation.)

Let H_0 and J_0 be real numbers satisfying $0 < |J_0| < H_0$. The Lagrangian submanifold

$$H = H_0, \quad J = J_0 \tag{27}$$

is a torus foliated by trajectories with initial velocity (ξ_0, η_0) on $x = 0$, with $\xi_0 > 0$, $\xi_0^2 + \eta_0^2 = H_0$ and $J_0 = \eta_0$. These trajectories are tangent to the caustic

$$x = \frac{\xi_0^2}{\eta_0^2} = \frac{H_0^2 - J_0^2}{J_0^2},$$

depicted as a dotted line in the figure below and in Section 1.

Fix $H = 1$, so that $\xi_0^2 + \eta_0^2 = 1$. Denote by $\gamma(t) = (x(t), y(t), \xi(t), \eta(t))$ the trajectory given by Proposition 2 on the interval $-T/2 \leq t \leq T/2$. Define by γ_1 the loop (whose projection onto $\mathbb{R}^+ \times S^1$ is pictured in bold in the figure

below) that follows γ on its two curved portions and then the segment of the caustic $x = \xi_0^2/\eta_0^2$, with $\xi = 0$, $\eta = \eta_0$ (oriented by $\dot{y} < 0$) going from $\gamma(T/2)$ to $\gamma(-T/2)$.



Define by γ_2 the loop that follows the caustic $x = \xi_0^2/\eta_0^2$, $0 \leq y \leq 2\pi$, $\xi = 0$, and $\eta = \eta_0$, with orientation $\dot{y} > 0$.

The loops γ_1 and γ_2 are a basis for the homology of the torus $H = 1$, $J = \eta_0$, and the action coordinates are defined by

$$I_1 = \int_{\gamma_1} \xi dx + \eta dy, \quad I_2 = \int_{\gamma_2} \xi dx + \eta dy.$$

Since $\eta(t) = \eta_0$ is constant, $T = 4\xi_0/\eta_0^2$, and $H = H_0 = \xi_0^2 + \eta_0^2$, we have

$$\begin{aligned} I_1 &= \int_{\gamma_1} \xi dx = \int_{\gamma} \xi dx \\ &= 2 \int_0^{T/2} (\xi_0 - \eta_0^2 t/2)^2 dt \\ &= \frac{4\xi_0^3}{3\eta_0^2} = \frac{4}{3}(H_0^2 - \eta_0^2)^{3/2}\eta_0^{-2} \end{aligned}$$

Moreover,

$$I_2 = \int_{\gamma_2} \xi dx + \eta dy = \int_{\gamma_2} \eta dy = 2\pi\eta_0$$

More generally, by homogeneity,

$$I_1 = \frac{4}{3}(H^2 - J^2)^{3/2}J^{-2}; \quad I_2 = 2\pi J.$$

Thus H and J can be written as functions of the action variables,

$$H^2 = J^2 + \left(\frac{3}{4}I_1 J^2\right)^{2/3} = \frac{I_2^2}{4\pi^2} + \left(\frac{3I_1 I_2^2}{16\pi^2}\right)^{2/3}; \quad J = \frac{I_2}{2\pi}.$$

Define $\Lambda(m, n)$ as the values of H^2 when the action variables belong to the integer lattice $(I_1, I_2) = 2\pi(m, n)$, $(m, n) \in \mathbb{Z}^2$. (Since $I_1 > 0$, we consider only $m > 0$.) Then

$$\Lambda(m, n) = n^2 + (3\pi/2)^{2/3}m^{2/3}n^{4/3}.$$

The Bohr-Sommerfeld energy levels of H are defined as the values of H on the lattice, namely $\sqrt{\Lambda(m, n)}$.

Proposition 10. *In any sector of the form $0 < c_1 \leq |n|/m \leq c_2$, the eigenvalues $\lambda(m, n)$ of the Friedlander operator satisfy*

$$\sqrt{\lambda(m, n)} = \sqrt{\Lambda(m, n)} + O(1), \quad (m, n) \rightarrow \infty$$

The proposition is an immediate consequence of the formula for $\mathcal{L}(m, n)$ and the fact that

$$\lambda(m, n) = n^2 + n^{4/3}\tau(m),$$

with $\tau(m) = (3\pi m/2)^{2/3} + O(1)$, $m = 1, 2, \dots$ (see Proposition 6).

Next, we show that the phases of the eigenfunctions are equal, asymptotically, to the generating function of the canonical relation of the Lagrangian submanifold (27). Indeed, if $\rho(x, y)$ is the generating function, then by definition, the Lagrangian is the graph of $(x, y) \mapsto (\rho_x, \rho_y)$:

$$\rho_x^2 + (1+x)\rho_y^2 = H_0^2; \quad \rho_y = J_0.$$

Thus,

$$\rho_x = \pm \sqrt{H_0^2 - J_0^2 - xJ_0^2}; \quad 0 \leq x \leq \frac{H_0^2 - J_0^2}{J_0^2},$$

and

$$\rho(x, y) = \pm \frac{2(H_0^2 - J_0^2 - xJ_0^2)^{3/2}}{3J_0^2} + J_0 y$$

Notice that the upper limit of the range of x is exactly the caustic

$$x = \frac{H_0^2 - J_0^2}{J_0^2},$$

so that the graph over this interval in x (and all y) covers the whole Lagrangian submanifold.

Now let the submanifold be given by $(I_1, I_2) = 2\pi(m, n)$. Then

$$H_0^2 = n^2 + \left(\frac{3}{2}\pi mn^2\right)^{3/2}; \quad J_0 = n,$$

and we have two generating functions

$$\rho_{\pm}(x, y) = \pm \frac{2}{3} \left(\left(\frac{3\pi}{2}m\right)^{2/3} - n^{2/3}x \right)^{3/2} + ny.$$

To compare this with the phase of $\varphi_{m,n}$, recall that for $t_m > n^{2/3}x$,

$$\begin{aligned} \varphi_{m,n}(x, y) &= \text{Ai}(n^{2/3}x - t_m)e^{iny} \\ &\sim \frac{1}{2}\pi^{-1/2}|n^{2/3}x - t_m|^{-1/4} \sin\left(\frac{2}{3}(t_m - n^{2/3}x)^{3/2} + \pi/4\right)e^{iny}. \end{aligned}$$

(See Proposition 1 and Appendix A.) Thus $\varphi_{m,n}$ is a sum of two terms with phases

$$\pm \left[\frac{2}{3}(t_m - n^{2/3}x)^{3/2} + \pi/4 \right] - \pi/2 + ny = \rho_{\pm}(x, y) + O(1),$$

as $(m, n) \rightarrow \infty$ in sectors of the form $0 < c_1 < |n|/m < c_2$ (since $t_m = (3\pi m/2)^{2/3}(1 + O(1/m))$).

Appendix A: Airy functions

An Airy function is a solution $y(t)$ of the differential equation $y'' = ty$. The classical properties of Airy functions are treated in [AS], Section 10.4 and [H], Section 7.6, and reviewed briefly here.

The *Airy function* (or *Airy integral*), denoted Ai , was introduced in 1838 by G.B. Airy in the remarkable paper [Ai] in order to describe the intensity of the light near a caustic. This allowed him to propose a good theory for the rainbow. He defined the function Ai by the oscillatory integral

$$\text{Ai}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(sx+x^3/3)} dx \quad (28)$$

In other words, Ai is the inverse Fourier transform of $x \mapsto e^{ix^3/3}$.

The Airy function $\text{Ai}(t)$ is positive for $t \geq 0$ and decays exponentially as $t \rightarrow +\infty$; it is oscillating for $t < 0$ with an expansion, given by the method of stationary phase, of the form

$$\text{Ai}(t) = \pi^{-1/2}|t|^{-1/4} \left[\sin \left(\zeta + \frac{\pi}{4} \right) + \text{Re} (e^{i\zeta} \alpha(\zeta)) \right], \quad t < 0,$$

with $\zeta = \frac{2}{3}|t|^{3/2}$ and α a classical symbol of order -1 . Differentiating, we find

$$\text{Ai}'(t) = -\pi^{-\frac{1}{2}}|t|^{1/4} \left[\cos \left(\zeta + \frac{\pi}{4} \right) + \text{Re} (e^{i\zeta} \beta(\zeta)) \right], \quad t < 0,$$

with β a classical symbol of order -1 .

The function Ai has infinitely many real negative zeros

$$\dots < -t_m < \dots < -t_1 < 0.$$

To derive the asymptotic formula for t_m , denote

$$f(s) = is^{1/3}\text{Ai}(-s^{2/3}) - \text{Ai}'(-s^{2/3}), \quad s > 0.$$

The foregoing asymptotic expansions can be written

$$f(s) = \frac{1}{\sqrt{\pi}} s^{1/6} e^{i(2s/3+\pi/4)+c(s)},$$

where c is a complex valued classical symbol of order -1 . Define

$$\theta(s) = 2s/3 + \pi/4 + \operatorname{Im} c(s) = \arg f(s),$$

$\operatorname{Im} f(s) > 0$ for all $0 < s < t_1^{3/2}$, and specify the branch of θ by $0 < \theta(s) < \pi$ in that range. Thus θ is a classical symbol of order 1 with principal part $2s/3$ as $s \rightarrow \infty$, and

$$\tan \theta(s) = -\frac{s^{1/3} \operatorname{Ai}(-s^{2/3})}{\operatorname{Ai}'(-s^{2/3})}, \quad s > 0. \quad (29)$$

Proposition 11. *Let θ be as above. Then $\theta'(s) > 0$ for all $s > 1/4$, and there is an absolute constant $c_1 < 1$ such that*

$$\tau(\xi) = [\theta^{-1}(\pi\xi)]^{2/3} > 0, \quad \text{for } \xi \geq c_1,$$

is well defined; $\tau \in S_{cl}^{2/3}(\mathbb{R}_+)$ with principal term $(3\pi\xi/2)^{2/3}$. Finally,

$$t_m = \tau(m), \quad m = 1, 2, \dots,$$

where $0 > -t_1 > -t_2 > \dots$ are the zeros of Ai .

Proof.

$$\theta'(s) = (1/3|f'|^2)[2s^{2/3}a^2 + 2b^2 - s^{-2/3}ab], \quad a = \operatorname{Ai}(-s^{2/3}), \quad b = \operatorname{Ai}'(-s^{2/3}).$$

Therefore, $\theta'(s) > 0$ for $s > 1/4$. Because θ is increasing, the zeros of Ai are given by

$$\theta(t_m^{3/2}) = m\pi, \quad m = 1, 2, \dots$$

Let $\theta(1/4) = \theta_0$. $0 < \theta_0 < \theta(t_1^{3/2}) = \pi$ because $t_1^{3/2} \approx (2.33)^{3/2} > 1/4$. The inverse function σ of θ is a classical symbol of order 1 on $[\theta_0, \infty)$. Then $\tau(\xi) = \sigma(\pi\xi)^{2/3}$ is defined for all $\xi \geq \theta_0/\pi$ and satisfies all the properties of the proposition. \square

Now we turn to proof of Proposition 1 describing the spectrum of the Friedlander operator L . Begin by noting that if L is restricted to functions $f(x)$ of x alone, it becomes

$$Lf(x) = -f''(x),$$

which has completely continuous spectrum on $L^2([0, \infty))$. Next, consider φ in the orthogonal complement,

$$\{\varphi \in L^2(M) : \int_{\mathbb{R}/2\pi\mathbb{Z}} \varphi(x, y) dy = 0\}$$

By separation of variables, the eigenfunctions are of the form $f(x)e^{iny}$, where n is a non-zero integer, and f satisfies

$$L_n f := (-\partial_x^2 + n^2(1+x))f = -\lambda f \quad (30)$$

For $n \neq 0$, the operator L_n acting on functions satisfying $f(0) = 0$ is self-adjoint and has compact resolvent, so a complete system of eigenfunctions is obtained by solving (30) for λ and f such that $f(0) = 0$ and $f \in L^2([0, \infty))$. Making the change of variables

$$f(x) = A(n^{2/3}x - t),$$

equation (30) becomes

$$n^{4/3}(-A''(s) + sA(s)) = (\lambda - n^{4/3}t - n^2)A(s).$$

Choose t so that $\lambda = n^2 + n^{4/3}t$, then the equation is the Airy equation

$$-A''(s) + sA(s) = 0 \quad (31)$$

Since f is in $L^2([0, \infty))$, the same is true of A . Up to a constant multiple, there is a unique such function A , the Airy function Ai . The Dirichlet condition $f(0) = 0$ implies $\text{Ai}(-t) = 0$, and the roots of this equation are $0 < t_1 < t_2 < \dots$. Hence the formulas in the proposition for the eigenfunctions and eigenvalues are confirmed.

Appendix B: Semiclassical properties of eigenfunctions and eigenvalues for completely integrable systems

Let X be an n -dimensional manifold and P_1, \dots, P_n be commuting self-adjoint zero order semiclassical pseudodifferential operators with leading symbols p_1, \dots, p_n . In Section 5 we considered the example $X = \mathbb{R}^+ \times S^1$,

$$P_1 = \hbar^2(\partial_x^2 + (1+x)\partial_y^2); \quad P_2 = i\hbar\partial_y.$$

(These operators are of order zero with respect to ∂_x, ∂_y and multiplication by \hbar .)

We seek the joint eigenfunctions $\varphi(x)$, $x \in X$, of P_j in the form

$$\varphi(x) = a(x, \hbar)e^{i\rho(x)/\hbar}, \quad a(x, \hbar) = a_0(x) + \hbar a_1(x) + \hbar^2 a_2(x) + \cdots, \quad (32)$$

solving asymptotically as $\hbar \rightarrow 0$

$$P_j \varphi = \lambda_j(\hbar) \varphi, \quad j = 1, \dots, n, \quad \lambda_j(\hbar) = \lambda_j(0) + \hbar \lambda_{j,1} + \hbar^2 \lambda_{j,2} + \cdots.$$

Let Λ_ρ be the graph of $d\rho$ in T^*X . Λ_ρ is a Lagrangian submanifold and determines ρ up to an additive constant; ρ is known as the *generating function* of Λ_ρ . Setting $\hbar = 0$, one finds the *eikonal equation*, the first of the hierarchy of equations for a , ρ , and $\lambda_j(\hbar)$:

$$p_j|_{\Lambda_\rho} = \lambda_j(0). \quad (33)$$

Let us confine ourselves to the sequence $\hbar = 1/N$ for integers $N \rightarrow \infty$. Then (32) defines φ provided ρ is well-defined modulo $2\pi\mathbb{Z}$. To see what this entails, denote by κ the projection, $\Lambda_\rho \rightarrow X$, and by ι the inclusion, $\Lambda_\rho \rightarrow T^*X$, and denote

$$\alpha = \sum_{j=1}^n \xi_j dx_j,$$

the canonical 1-form on T^*X . The fact that Λ_ρ is the graph of $d\rho$ implies

$$d\kappa^* \rho = \iota^* \alpha. \quad (34)$$

If $(\lambda_1(0), \dots, \lambda_n(0))$ is a regular value of (p_1, \dots, p_n) and Λ_ρ is compact, then Λ_ρ is a torus of dimension n . Thus if $\gamma_1, \dots, \gamma_n$ is a basis of the first homology group of Λ_ρ , then (34) implies

Proposition 12. *ρ is well-defined modulo $2\pi\mathbb{Z}$ if and only if*

$$I_j := \int_{\gamma_j} \alpha \in 2\pi\mathbb{Z}, \quad j = 1, 2, \dots, n.$$

The functions I_j are known as action coordinates on T^*X . Let $\mathbf{m} \in \mathbb{Z}^n$. In the generic case in which

$$(I_1, \dots, I_n) = 2\pi\mathbf{m}$$

defines an n -dimensional Lagrangian torus in T^*X , the torus is known as a Bohr-Sommerfeld surface, and we will denote it by $S(\mathbf{m})$.

Proposition 12 says that the generating function $\rho_{\mathbf{m}}$ of $S(\mathbf{m})$ is well-defined modulo $2\pi\mathbb{Z}$. Thus one can follow the ansatz (32) to seek a joint eigenfunction φ whose phase function ρ equals $\rho_{\mathbf{m}}$ to first order as $\hbar \rightarrow 0$ and whose eigenvalue λ_j with respect P_j is given to first order by p_j evaluated on $S(\mathbf{m})$ as in (33). Under suitable further hypotheses, all but finitely many joint eigenfunctions are obtained in this way. Typically, φ and $S(\mathbf{m})$ exist for a cone of values of $\mathbf{m} \in \mathbb{Z}^n$ (see [CdV1]).

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