18.155 Problem Set 9

Due Wednesday, 12/10/08

Define the space $H^1(\Omega)$ as the space of all function u such that $u \in L^2(\Omega)$ and $\nabla u \in L^2(\Omega)$ with inner product

$$\langle u, v\rangle_1 = \int_\Omega [\nabla u(x) \cdot \nabla \bar{v}(x) + u(x)\bar{v}(x)]dx$$

and norm $\|u\|^2_{H^1(\Omega)} = \langle u, u\rangle_1$. Assume that a^{ij}, b^j and c are infinitely differentiable functions on $\mathbf{R}^n$ with a^{ij} a real symmetric matrix satisfying the ellipticity condition,

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq m|\xi|^2$$

for some constant $m > 0$ and all $\xi \in \mathbf{R}^n$. For u and v in $H^1(\Omega)$, define

$$Q(u, v) = \int_\Omega a^{ij}\frac{\partial u}{\partial x_i}\frac{\partial \bar{v}}{\partial x_j}dx + \int_\Omega b^j\frac{\partial u}{\partial x_i}\bar{v}dx + \int_\Omega cu\bar{v}dx$$

We also write

$$Lu(x) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a^{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_j b^j(x)\frac{\partial u}{\partial x_i} + c(x)u(x)$$

See Evans's book, Chapter 5 for a proof that for domains Ω with C^1 boundary, the space $H^1(\Omega)$ is the restriction to Ω of functions in $H^1(\mathbf{R}^n)$. Furthermore, $C^\infty(\bar{\Omega})$ functions are dense in $H^1(\Omega)$.

Recall that the closure of $C_c^\infty(\Omega)$ in this norm is the subspace $H_0^1(\Omega)$. This subspace can also be identified via the restriction theorem. Consider a function in $C^\infty(\bar{\Omega})$, then the restriction operator, $R : C^\infty(\bar{\Omega}) \to C(\partial\Omega)$ given by $Rf(x) = f(x)$ for $x \in \partial\Omega$ satisfies

$$\|Rf\|_{L^2(\partial\Omega, d\sigma)} \leq C\|f\|_{H^1(\Omega)}$$

where $d\sigma$ is surface measure on $\partial\Omega$. It follows that the mapping R extends to a bounded linear mapping on all of $H^1(\Omega)$ to $L^2(\partial\Omega, d\sigma)$. As proved in Evans's book, the subspace $H_0^1(\Omega)$ is the null space of this mapping.

1. We carry out here the application of Fredholm theory to general quadratic forms like Q. First we will restrict to the space $H_0^1(\Omega)$, which gives rise to the Dirichlet problem. (See Chapter 6 of Evans's book if you need further hints.)

a) Denote $\langle u, v\rangle_0 = \int_\Omega u(x)\bar{v}(x)dx$. Define $Q_\lambda(u, v) = Q(u, v) + \lambda\langle u, v\rangle_0$. Show that for λ sufficiently large (real) there is a constant $c > 0$ such that

$$|Q(u, u)| \geq c\int_\Omega [|\nabla u|^2 + |u|^2]dx$$

Note also that $|Q_\lambda(u, v)| \leq C\|u\|_{H^1}\|v\|_{H^1}$.

b) Show that for λ sufficiently large there is a bounded linear operator $T : L^2(\Omega) \to H_0^1(\Omega)$ such that

$$Q_\lambda(Tf, v) = \langle f, v\rangle_0 \quad \text{for all } v \in H_0^1(\Omega)$$

following these steps

i) The mapping $v \mapsto Q_\lambda(u, v)$ is $\bar{\mathbf{C}}$-linear on $H_0^1(\Omega)$, so by the Riesz representation theorem for the Hilbert space $H_0^1(\Omega)$, there is w such that

$$\langle w, v\rangle_1 = Q_\lambda(u, v) \quad \text{for all } v \in H_0^1(\Omega)$$

Show that $w = Au$ is a bounded linear mapping on $H_0^1(\Omega)$ and that A is invertible if λ is sufficiently large.

ii) Show that there is a bounded linear mapping $B : L^2(\Omega) \to H_0^1(\Omega)$ such that $\langle Bf, v\rangle_1 = \langle f, v\rangle_0$ and show that the composition $T = A^{-1}B$ does the job.

c) Define $Q^*(u, v) = \overline{Q(v, u)}$ and $Q^*_\lambda(u, v) = Q^*(u, v) + \lambda\langle u, v\rangle_0$. It follows from (b) that for sufficiently large λ, there is a bounded operator $S : L^2(\Omega) \to H_0^1(\Omega)$ such that

$$Q^*_\lambda(Sf, v) = \langle f, v\rangle_0 \quad \text{for all } v \in H_0^1(\Omega)$$

Show that $S = T^*$ on $L^2(\Omega)$, that is, for all f and g in $L^2(\Omega)$, $\langle Tf, g\rangle_0 = \langle f, Sg\rangle_0$.

d) Recall that for $f \in L^2$, we say $Lu = f$ in the weak sense if $u \in H_0^1(\Omega)$ and $Q(u, v) = \langle f, v\rangle_0$ for all $v \in H_0^1(\Omega)$. Show that if u is a weak solution then $(I - \lambda T)u = Tf$ and conversely, if $u \in L^2(\Omega)$ satisfies $(I - \lambda T)u = Tf$, then $u \in H_0^1(\Omega)$ and u is a weak solution. In particular, if $u \in N(I - \lambda T)$, the null space of $I - \lambda T$ (in $L^2(\Omega)$), then $u \in H_0^1(\Omega)$ and $Lu = 0$ in the weak sense.

e) Denote by $N(L)$ the null space of weak solutions $v \in H_0^1(\Omega)$ to $Lv = 0$ in Ω, and similarly for L^*. (What is L^*?) Explain why $N(L)$ and $N(L^*)$ are finite dimensional. Show that if $f \in L^2(\Omega)$, the equation $Lu = f$ for $u \in H_0^1(\Omega)$ can be solved uniquely up to addition of an element of $N(L)$ if and only if $\langle f, v\rangle_0 = 0$ for all $v \in N(L^*)$.

Remark: Assuming the regularity theory (proved in lecture), we know that if the boundary of Ω is smooth and the right side f is smooth, then the function u is smooth. Thus the solution $u \in H_0^1(\Omega)$ is infinitely differentiable. The restriction theorem implies $Ru \in L^2(d\sigma)$ is zero. It follows that $u(x) = 0$ for $x \in \partial\Omega$.

f) Show that a smooth function $u \in C^\infty(\bar{\Omega})$ such that $u(x) = 0$ on $\partial\Omega$ belongs to $H_0^1(\Omega)$.

2. Prove the Poincaré inequality,

$$\inf_a \int_Q |f(x) - a|^2 dx \le \int_Q |\nabla f(x)|^2 dx$$

for the unit cube $Q = [0, 1]^n$ as follows. We will assume that $f \in C^\infty(\bar{Q})$, although this estimate is valid for any distribution (vacuous unless the right side is finite).

a) Do the case $n = 1$ using $a = f(0)$ and the fundamental theorem of calculus.

b) The rest of the approach shows that whatever constant you get the in dimension 1 works in all dimensions. Note that

$$\inf_a \int_0^1 |f(x) - a|^2 dx = \int_0^1 |f(x) - a_1|^2 dx$$

where $a_1 = \displaystyle\int_0^1 f(x)dx$, the average. Denote

$$f_1(x) = \int_0^1 f(y, x_2, x_3, \ldots, x_n)dy; \quad f_{12}(x) = \int_0^1 \int_0^1 f(y_1, y_2, x_3, \ldots, x_n)dy_1dy_2;$$

Note that f_1 is independent of x_1 and f_{12} is independent of x_1 and of x_2. Use the 1-dimensional inequality to show ($x = (x_1, x_2, \ldots, x_n)$)

$$\int_0^1 |f(x) - f_1(x)|^2 dx_1 \le \int_0^1 \left|\frac{\partial f}{\partial x_1}\right|^2 dx_1$$

With a little extra work, show that

$$\int_0^1 \int_0^1 |f_1(x) - f_{12}(x)|^2 dx_1dx_2 \le \int_0^1 \int_0^1 \left|\frac{\partial f}{\partial x_2}\right|^2 dx_1dx_2$$

Finally, observe that $f - f_1$, $f_1 - f_{12}$, etc, are perpendicular to each other to get the desired inequality with constant 1.

Remark: This method does not give the best constant, which is $1/\pi^2$. The best constant can be derived in one variable by ordinary Fourier analysis or directly from the calculus of variations, that is the Neumann problem for the operator $L = -(d/dx)^2$ on $[0, 1]$ and the extremal function $\cos(\pi x)$. The n-variable case can also be derived from n-dimensional Fourier analysis. But what (b) shows is that for any product of domains, the worst of the constants in each factor is the constant that works for the product.

c) Prove that for any bounded C^1 domain Ω there is a constant C such that

$$\inf_a \int_\Omega |f(x) - a|^2 dx \le C \int_\Omega |\nabla f(x)|^2 dx$$

Hint: Make a C^1 change of variables so that the boundary is given locally by a hyperplane. Then use the Poincaré inequality for a finite collection of cubes covering the whole region Ω. Make cubes have significant overlap adjacent ones so that there is an overlapping chain to one central cube.

3. Do the same thing as in Problem 1 with the space $H^1(\Omega)$ replacing the space $H_0^1(\Omega)$. This is practically the same as before, requiring only the Poincaré inequality of Problem 2 in place of the baby Poincaré inequality for compactly supported functions in Ω. The boundary conditions are no longer Dirichlet conditions, but are what are known as Neumann conditions. It does not appear as if u has a boundary condition, but this is hidden in the variational formulation. The fact that

$$Q(u, v) = \langle f, v\rangle_0$$

is true for $v \in H_0^1(\Omega)$ shows that $Lu = f$ in the weak sense in the interior Ω. Now suppose that $u \in C^\infty(\bar{\Omega})$. Show that the fact that the equation also holds for $v \in H^1(\Omega)$ implies an equation for u on the boundary. Do this first for $L = -\Delta$, then the general variable coefficient case. What you should get is a normal derivative (Neumann type) condition with respect to a direction known as the conormal in the general case.

4. The Harnack inequality. Consider an elliptic operator L in non-divergence form with C^∞ coefficients,

$$Lu = -a^{ij}u_{ij} = -\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}$$

We will be using subscripts i, j, etc, for differentiation with respect to the variables x_i, x_j. We will also use the Einstein summation convention, in which repeated indices are summed. This will save you a lot of writing!

Theorem: Let $u \in C^\infty(B_2)$ be a nonnegative function. Suppose also that $Lu = 0$ in B_2. Then

$$\sup_{B_1} u \le C \inf_{B_1} u$$

where the constant C depends only on the ellipticity constant of L and bounds on a few derivatives of a^{ij}. (This is a special case of the theorem for parabolic equations in Evans's text in Section 7.1.)

There are several remarkable, much deeper theorems in the case of minimal smoothness, in which the distinction between divergence and nondivergence form becomes very important. These more general theorems are central to the theory of nonlinear elliptic equations. Other forms of the Harnack inequality are important in geometry, and the proof below is more closely tied to the geometric variants.

Define $v = \log(u + \epsilon)$ for some $\epsilon > 0$, and $w = Lv$ (At the very end of the problem, remember to let ϵ tend to 0. We need to add ϵ so that $u + \epsilon > 0$ and v is a smooth function.) Note that a corollary of the Harnack inequality is the strict minimum principle for u, namely, if u achieves its minimum, 0, then u is identically zero. The Harnack inequality is a quantitative form of the (strict) maximum or minimum principle.

a) Show that $w = a^{ij}v_iv_j$ and $c_1|Dv|^2 \le w \le c_2|D^2v|$ for some positive constants c_1 and c_2. Here and below we are using the notations $Dv = \nabla v$, the gradient, and D^2v is the Hessian matrix. $|D^2v|$ is any norm on the n^2 dimensional vector space of matrices such as the square root of the sum of squares of entries.

b) Show that for some positive constants c_i,

$$Lw + b^k w_k \le -c_1|D^2v|^2 + c_2|Dv|^2 + c_3, \quad \text{with} \ \ b^k = 2a^{k\ell}v_\ell$$

c) Let $\eta \in C^\infty(B_2)$ be a smooth bump function $0 \le \eta \le 1$ with $\eta = 1$ on B_1. Let x^0 be a point at which $\eta^4 w$ attains its maximum. Explain why $D(\eta^4 w) = 0$ and $L(\eta^4 w) \ge 0$ at x^0.

d) Show that at x^0, $\eta^4|Dw| \le C\eta^3 w$ and $\eta^4 Lw \ge -C\eta^2 w$.

e) Deduce that at x^0,

$$\eta^4|D^2v|^2 \le C(\eta^4 + \eta^4|Dv|^2 + \eta^3|Dv|w + \eta^2 w) \le C'(\eta^4 + \eta^3|D^2v|^{3/2} + \eta^2|D^2v|)$$

f) Show that for every $\epsilon > 0$, $z^3 + z^2 \le \epsilon z^4 + C(\epsilon)$ for some $C = C(\epsilon)$, and deduce that

$$\eta^3|D^2v|^{3/2} + \eta^2|D^2v| \le \epsilon\eta^4|D^2v|^2 + C(\epsilon)$$

g) Show that there is a constant C with the dependence of the theorem such that at x^0

$$\eta^4|D^2v|^2 \le C$$

h) Recall how x^0 was defined and deduce that there is a constant C with the dependence of the theorem for which

$$\sup_{B_1} |Dv| \le C$$

Deduce the Harnack inequality from this.