18.155 Problem Set 8; Part II

Due Wednesday, 12/03/08

1. Consider a mapping $Q : H \times H \to \mathbf{C}$ that is sesquilinear, i.e., linear in x and conjugate linear in y. Suppose that $Q(v,u)$ is the conjugate of $Q(u,v)$, and $Q(u,u) \geq 0$ (positive semidefinite). Show that $|Q(u,v)| \leq Q(u,u)^{1/2}Q(v,v)^{1/2}$

2. a) Show that if x_j is a norm-bounded sequence in a Hilbert space H, then x_j has a weakly convergent subsequence. (Note that the closure of the span of the sequence x_j is a separable Hilbert space, so that we may as well assume that H has a countable orthonormal basis e_n, $n = 1, 2, \ldots$.) Hint: Explain why there is a subsequence y_ℓ for which $\langle y_\ell, e_n \rangle \to a_n$ as $\ell \to \infty$ for every n. Then show that $\sum |a_n|^2 < \infty$ and that $\langle y_\ell, z \rangle \to \langle y, z \rangle$ as $\ell \to \infty$ for every $z \in H$ where y is what vector?

b) Show that if S is a closed subspace of H, then there exists a unique bounded linear mapping $P : H \to H$ satisfying $Px = x$ for all $x \in S$, $P^* = P$, and $Px = 0$ for all $x \in S^\perp$.

c) Prove the Riesz representation theorem, namely, that for every bounded linear functional $\Lambda : H \to \mathbf{C}$ there is a unique $u \in H$ such that $\Lambda(v) = \langle v, u \rangle$. (The right side is the inner product on the Hilbert space.)

d) Note that $H^1(\mathbf{R}^n)$ is a Hilbert space with inner product,

$$\langle f, g \rangle = \int_{\mathbf{R}^n} \nabla f \cdot \nabla \bar{g} + f\bar{g}\, dx$$

Use part (c) to show that the dual space is the space of all distributions of the form

$$f_0 + \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j}$$

where $f_j \in L^2(\mathbf{R}^n)$ for $j = 0, 1, \ldots, n$. Don't use the Fourier transform. (There's another proof using the Fourier transform and the fact that the dual of L^2 is L^2.)

e) Let Ω be an open subset of $\mathbf{R}^n$ Show the dual space of $H^1_0(\Omega)$ is the space similar to (d) with $f_j \in L^2(\Omega)$.

3. Hölder's inequality. a) Show that $u^\theta v^{1-\theta} \leq \theta u + (1-\theta)v$ for all $u > 0$, $v > 0$ and $0 \leq \theta \leq 1$. Hint: Show the left side is a convex function of of θ.

b) Substitute $u^\theta = |Af|$, $v^{1-\theta} = |Ag|$ with $1/p = \theta$ and $1/p' = 1 - \theta$. Consider $f = f(x)$ and $g = g(x)$ and integrate with respect to measure $d\mu(x)$. Optimize in A to get

$$\int_X |fg| d\mu \leq \|f\|_p \|g\|_{p'}$$

where

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}$$

for $1 \leq p < \infty$ (and $\|f\|_\infty = \text{ess sup}|f(x)|$).

c) (converse inequality) Show that if $1 \leq p \leq \infty$ is fixed, f is measurable, and

$$\sup \Re \int fg d\mu \leq C$$

for all $\|g\|_p \leq 1$, then $f \in L^{p'}$ and $\|f\|_{p'} \leq C$. (Use a slightly different argument at the endpoints $p = 1$ and $p = \infty$.)