18.155 Problem Set 8; Part I

Due Wednesday, 12/03/08

1. Let Ω be a bounded, open subset of $\mathbf{R}^n$. Define $H_0^s(\Omega)$ as the closure in the $H^s(\mathbf{R}^n)$ norm of functions in $C_c^\infty(\Omega)$.

a) Show that for $s \geq 0$,

$$\|\hat{f}\|_{C^1(\mathbf{R}^n)} := \max_{\xi \in \mathbf{R}^n}(|\hat{f}(\xi)| + |\nabla \hat{f}(\xi)|) \leq C\|f\|_{H^s(\mathbf{R}^n)}$$

for every $f \in C_c^\infty(\Omega)$, and conclude that $\hat{f}$ is a well-defined function in $C^1(\mathbf{R}^n)$ (with the same uniform bound as for the dense subclass) for every $f \in H_0^s(\Omega)$. (The restriction $s \geq 0$ is not necessary but makes the proof easier. What is crucial is that Ω is bounded.)

b) Show that the inclusion $H_0^s(\Omega) \subset L^2(\Omega)$ is compact for $s > 0$.

2. For $h \in \mathbf{R}^n$, $h \neq 0$, denote $D^h f(x) = (f(x+h) - f(x))/|h|$.

a) Let $f \in L^1_{loc}(\mathbf{R}^n)$. Show that for $t > 0$, $|h| = 1$, $D^{th}f$ tends to $h \cdot \nabla f$ in the sense of distributions as $t \to 0$.

b) Show that if $f \in L^p(\mathbf{R}^n)$, $1 < p \leq \infty$ and

$$\sup_{0<|h|\leq 1} \|D^h f\|_{L^p(\mathbf{R}^n)} \leq C$$

then $f \in W^{1,p}(\mathbf{R}^n)$, the space of all functions $f \in L^p(\mathbf{R}^n)$ such that the distributional derivative $\nabla f \in L^p(\mathbf{R}^n)$. (Use the characterization of L^p norm, $1 < p \leq \infty$ via duality.)

c) Show that (b) fails for $p = 1$.

d) Prove a converse. If $1 \leq p < \infty$, and $f \in W^{1,p}(\mathbf{R}^n)$, then

$$\sup_{|h|>0} \|D^h f\|_{L^p(\mathbf{R}^n)} \leq \|\nabla f\|_{L^p(\mathbf{R}^n)}$$

This assertion is also true for $p = \infty$ as you will demonstrate in Problem 3. (Hint: Assume, in addition, $f \in C^1(\mathbf{R}^n)$. In this case the estimate can be proved using the fundamental theorem of calculus and Minkowski's inequality. Moreover, one can approximate in a standard way. Let φ be a smooth, compactly supported function on $\mathbf{R}^n$ with

$$\int \varphi(x)dx = 1$$

and denote $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$. Show that

$$\|f * \varphi_\epsilon - f\|_{L^p} \to 0$$

for all $f \in L^p(\mathbf{R}^n)$, $1 \leq p < \infty$. For the sharp constant 1 in the inequality, you will also need to assume $\varphi \geq 0$.)

3. a) Show that for $f \in C^1$ and $p > n$,

$$\int_{B_1(x_0)} |f(x) - f(x_0)|dx \leq C \left(\int_{B_1(x_0)} |\nabla f(x)|^p dx \right)^{1/p}$$

The constant C depends only on p and n. Hint: Without loss of generality, let $x_0 = 0$ and apply the fundamental theorem of calculus to $g(t) = f(tx) - f(0)$, $0 \leq t \leq 1$.

b) Show by scaling that there is at most one value of α, $0 < \alpha < 1$ depending on p and n for which

$$\max_{|x-y|\le r} |f(x) - f(y)| \le C r^\alpha \|\nabla f\|_{L^p(B_{2r}(x))}$$

for all $f \in C^1$, and find the formula for α. Then prove that the inequality is true using part (a).

c) Show that for $p > n$, $f \in C^1(\mathbf{R}^n)$,

$$\max |f| \le C(\|\nabla f\|_{L^p(\mathbf{R}^n)} + \|f\|_{L^p(\mathbf{R}^n)})$$

d) Deduce that $W^{1,p}(\mathbf{R}^n) \subset C^\alpha(\mathbf{R}^n)$ with α you found in part (b).

e) Show (by example) that $W^{1,\infty}(\mathbf{R}^n)$ is not a subset of $C^1(\mathbf{R}^n)$, On the other hand, show that the endpoint result corresponding to 2(b) and (d) above is valid with an exact control on the constant. A distribution f has a continuous representative satisfying

$$|f(x) - f(y)| \le A|x - y|$$

if and only if the distributional derivative of f, satisfies $\nabla f \in L^\infty(\mathbf{R}^n)$ and $\|\nabla f\|_{L^\infty} \le A$. In particular, the class $W^{1,\infty}(\mathbf{R}^n)$ is the same as the class of Lipschitz functions, that is bounded functions on $\mathbf{R}^n$ satisfying

$$|f(x) - f(y)| \le C|x - y|$$

Hint: Show that

$$\|f * \varphi_\epsilon - f\|_{L^\infty} \to 0,$$

holds for all $f \in W^{1,\infty}(\mathbf{R}^n)$. The difficulty with the argument you used in Problem 2 is that this property fails for some $f \in L^\infty(\mathbf{R}^n)$.