18.155 Problem Set 7; Parts I and II

Due Wednesday, 11/12/08

In Problem 1, we recapitulate the problems from PS6 so that you can finish them.

1. Recall from PS6 that you showed that for $z \in \mathbf{C}$ and $n = 1, 2, \ldots$

$$(x \pm i0)^z = x_+^z + e^{\pm i\pi z} x_-^z \quad (z \neq -1,\ -2,\ \ldots)$$

$$(x \pm i0)^{-n-1} = \frac{(-1)^n}{n!}\left(\frac{d}{dx}\right)^n (x \pm i0)^{-1} = \frac{(-1)^n}{n!}\left(\frac{d}{dx}\right)^n (\text{p.v.}\frac{1}{x} \mp i\pi\delta)$$

a) Consider the distributions $u_z(x) = (x+i0)^z$, $h_z(x) = x_+^z/\Gamma(z+1)$. We proved in PS6 that u_z is defined for all $z \in \mathbf{C}$. On PS6 you showed that $h_z(\varphi)$ is a meromorphic function of z, defined except possibly at $z = -1, -2, \ldots$. Prove that again by showing that h_z initially defined for $\Re z > -1$ is extended to all values of z by the formula

$$h_z = (d/dx) h_{z+1}$$

In particular confirm that (because we have put in an extra gamma function factor in the denominator with a pole at each negative integer, the expression is well-defined at these values as well. In fact, the limit

$$h_{-k} = \lim_{z \to -k} h_z = \delta^{(k-1)}$$

where $\delta^{(k-1)}$ is the $k-1$st derivative of the delta function.

b) Find the formula for the Fourier transform of $(x + i0)^z$ by computing the coefficient $c(z)$ in the formula

$$\hat{u}_z = c(z) h_{-1-z}$$

for all $z \in \mathbf{C}$, using the procedure outlined as follows. For complex numbers satisfying $\Re z > 0$, the gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Change variables by $y = at$ with $a > 0$, then use analytic continuation to extend the formula to $a - ix$ for $x \in \mathbf{R}$. Then take the limit as $a \to 0$. Finally, replace z by $-z$. This confirms the formula in the case $\Re z < 0$. Finally, show that for all $f \in \mathcal{S}(\mathbf{R})$ the expressions

$$u_z(\hat{\varphi}) \quad \text{and} \quad c(z) h_{-1-z}(\varphi)$$

are analytic in z and hence that the equation between them extends to all values of z. Check your formula for $c(z)$ by looking at the cases $z = 1, 2, \cdots$.

c) Recall from PS6 that the distribution E on $\mathbf{R} \times \mathbf{R}^n$ defined in terms of its partial Fourier transform by $\mathcal{F}_2 E = H(t)\sin(t|\xi|)/|\xi|$, satisfies

$$\Box E = \delta$$

where $\Box$ denotes the wave operator

$$\Box = \frac{\partial^2}{\partial t^2} - \Delta_x$$

Show that E is homogeneous of degree $1 - n$. Denote

$$F_1 = h_{(1-n)/2}(t^2 - |x|^2) H(t)$$

Calculate $\mathcal{F}_2 F_1(t, 0)$ for $t > 0$. Show that

$$F_1 = cE$$

and calculate the constant c. Here is the first half of the argument, which you should repeat in your own words. You showed on PS6 that F_1 is well-defined as a homogeneous distribution of degree $1-n$, and satisfies $\Box F_1 = 0$ outside the origin and hence satisfies $\Box F_1 = c\delta$ in all of $\mathbf{R}^{n+1}$. F_1 is obviously rotation invariant. Moreover, on PS6 you found that all homogeneous distributions F of degree $1-n$ that are rotation-invariant in x and satisfy $\Box F = c\delta$ form a three-dimensional family. Since F_1 is the only member of that family that is supported in $t \geq 0$, it must be a multiple of E.

Further hints: To evaluate the constant it helps to know that $\Gamma(z+1) = z\Gamma(z)$ and to use the relationship $(d/dx)h_z = h_{z-1}$.

d) Let $f : \mathbf{R}^N \to \mathbf{R}$ be a smooth function such that $\nabla f(p) \neq 0$ and $f(p) = 0$. Then for $z \neq 0, 1, 2, \ldots$,

$$WF(u_z(f)) \cap \{p\} \times S^n = \{(p, \nabla f(p)/|\nabla f(p)|)\}$$

What is the analogous result with $u_z(y)$ replaced by $(y - i0)^z$?

Hint: The wave front set projects onto the singular support. Why is p in the singular support of $u_z(f)$? Use the inverse Fourier transform representation of u_z to represent the Fourier transform of $\psi u_z(f)\psi$ (for a suitable cutoff function ψ) as an oscillatory integral (See 3(c)). This can be used to show that every direction besides the direction of $\nabla f(p)$ is not in the wave front set at p.

e) Explain why if $WF(v_1) \cap WF(v_2) = \emptyset$, then $WF(v_1 + v_2) = WF(v_1) \cup WF(v_2)$. Deduce from this and from part (d) the wave front sets of all linear combinations of the distributions

$$F_1 = h_{(1-n)/2}(f)H(t), \quad F_2 = h_{(1-n)/2}(f)H(-t), \quad F_3 = u_{(1-n)/2}(f), \quad (f = t^2 - |x|^2)$$

at all points except the origin $(t, x) = (0, 0)$ $(n > 1)$. State the answer for $t > 0$ and $t < 0$ separately, and use the formulas from 1(a) above. (You may also distinguish n even and odd.)

f) Find the sign (according to whether $n > 1$ is even or odd) for which

$$G = (t^2 - |x|^2 + i0)^{(1-n)/2}H(t) \pm (t^2 - |x|^2 - i0)^{(1-n)/2}H(-t)$$

is a homogeneous solution to $\Box G = c\delta$. Point to where in your proof you used the correct sign and where the argument would go wrong with the wrong sign. Identify the wave front set of G at points other than the origin $(0, 0)$.

g) Recall that we defined $G_\pm$ by

$$\mathcal{F}_2 G_\pm = e^{\pm it|\xi|}/|\xi|$$

Define

$$U_\pm = \delta(\tau^2 - |\xi|^2)H(\pm\tau)$$
$$V = (\tau^2 - |\xi|^2 + i0)^{-1}H(\tau) + (\tau^2 - |\xi|^2 - i0)^{-1}H(-\tau)$$
$$W = \text{p.v.}\frac{1}{\tau^2 - |\xi|^2}$$

Repeating PS6, identify $\hat{G}_\pm$ as a linear combination of $U_\pm$. Show that the inverse Fourier transform of U_+ is a linear combination of the three distributions in part (e). Recall that the projection of the scattering wave front set is the cone singular support of the Fourier tranform of a distribution: $\pi_2 WF_{sc}(w) = Css(\hat{w})$. Show that is follows from (e) that $\hat{G} = \alpha U_+$. Use this, the analogous result for $\tilde{G} = G((-t, -x)$, and the calculation of c in part (c) to evaluate the constant α.

h) Put together all these results to identify the Fourier transforms of F_j, $j = 1, 2, 3$ and G (for $n > 1$). Figure out, in particular, which distribution has partial Fourier transform $\cos(t|\xi|)/|\xi|$.

j) Use the fact that $WF(P(\partial)v) \subset WF(v)$ to figure out the wave front set of all of the F's and G's at the origin. It also helps to remember the projection property concerning the scattering wave front set, and the fact that the wave front set is closed. (Optional: find the scattering wave front set.)

2. We review here some elementary aspects of Fourier series. Recall that for a 2π-periodic function F on $\mathbf{R}$ or, equivalently, on $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$, the Fourier coefficients are defined by

$$a_k = a_k(F) = \frac{1}{2\pi}\int_0^{2\pi} F(x)e^{-inx}dx$$

Lemma (proved in part (c)) If $H \in C(\mathbf{T})$ and $a_k(H) = 0$ for all k, then $H = 0$.

a) Show that if $F \in C^2(\mathbf{T})$, then $|a_k| \le C/k^2$ so that the series

$$\sum_{k\in\mathbf{Z}} a_k e^{ikx}$$

converges uniformly. Deduce from the lemma that

$$F(x) = \sum_{k\in\mathbf{Z}} a_k e^{ikx}$$

b) The **Poisson summation formula** says that if $f \in \mathcal{S}$, then

$$\sum_{k\in\mathbf{Z}} f(2\pi k) = \frac{1}{2\pi}\sum_{n\in\mathbf{Z}} \hat{f}(n)$$

where $\hat{f}(\xi) = \displaystyle\int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$. Prove this by applying the Fourier inversion formula proved in part (a) to the periodized function $F(x) = \sum_k f(x+2\pi k)$. Write the Poisson summation formula as a statement about the Fourier transform on $\mathbf{R}$ in the sense of distributions of $\sum_K \delta(x-2\pi k)$.

c) Prove the lemma as follows.

i) For $0 \le r < 1$, define $P_r(x) = \displaystyle\sum_{k=-\infty}^{\infty} r^{|k|}e^{ikx}$. Show that

$$P_r(x) = \frac{1-r^2}{|1-re^{ix}|^2}$$

by summing a geometric series.

ii) Define convolution on $\mathbf{T}$ by

$$f * g(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-y)g(y)dy$$

Show that

$$\lim_{r\to 1} \max |F * P_r(x) - F(x)| = 0$$

for $F \in C(\mathbf{T})$.

iii) For $H \in C(\mathbf{T})$, show that $H * P_r(x) = \displaystyle\sum_{k\in\mathbf{Z}} a_k(H)r^{|k|}e^{ikx}$ and deduce the lemma.

3. Oscillatory Integrals. Recall the lemma of non-stationary phase from lecture:

$$\left|\int_{R^n} e^{it\varphi(x)}\psi(x)dx\right| \le C_j t^{-j}\|\psi\|_{C^j} \max_K \left(\frac{1}{|\nabla\varphi|} + \frac{1}{|\nabla\varphi|^{2j}}\right)$$

where ψ is supported in a compact subset $K_1 \subset K$ and C_j depends on $\|\varphi\|_{C^{j+1}(K)}$ and a positive lower bound for the distance from K_1 to the complement of K. All three parts of this exercise follow from this lemma.

The class S^m of symbols is defined as the collection of functions $a \in C^\infty(\mathbf{R}^n \times R^N)$ satisfying

$$|(\partial/\partial x)^\alpha(\partial/\partial\theta)^\beta a(x,\theta)| \le C_{\alpha,\beta}(1+|\theta|)^{m-|\beta|},$$

for all $(x,\theta) \in \mathbf{R}^n \times \mathbf{R}^N$.

Consider a real-valued phase function $\varphi \in C^\infty(\mathbf{R}^n \times (\mathbf{R}^N\backslash 0))$ such that $\varphi(x,t\theta) = t\varphi(x,\theta)$ for all $t > 0$ and $\nabla_{x,\theta}\varphi \ne 0$ (for $\theta \ne 0$.

a) Show that the oscillatory integral

$$I = \sum_{k=0}^{\infty}\int_{\mathbf{R}^N} \eta_k(\theta)a(x,\theta)e^{i\varphi(x,\theta)}d\theta = \sum_k I_k$$

converges in the sense of distributions. Here η_k is a dyadic partition of unity on $\mathbf{R}^N$:

$$\sum_k \eta_k(\theta) = 1$$

$supp\eta_0 \subset \{|\theta| < 1\}$, $supp\eta_k \subset 2^{k-2} < |\theta| < 2^{k+2}\}$, and $((\partial/\partial\theta)^\beta\eta_k(\theta)| \le C_\beta(1+|\theta|)^{-|\beta|}$. Notice that this means that the functions η_k belong to the symbol class S^0 with bounds independent of k. See Hörmander Vol I, Theorem 7.8.2. Oscillatory integrals are written (in)formally as

$$I = \int_{\mathbf{R}^N} a(x,\theta)e^{i\varphi(x,\theta)}d\theta$$

Hint: For a test function ψ, write each integral $I_k(\psi)$ as an integral in x and θ rescaling θ to unit size. Then apply the lemma in (x,θ)-space.

b) Show that $ss(I) \subset \{x \in \mathbf{R}^n : \nabla_\theta\varphi(x,\theta) = 0 \text{ for some} \theta \ne 0\}$. See Hörmander, Vol I, Theorem 7.8.3. Hint: apply the lemma to $I_k(x)$ for fixed x.

c) Show that $WF(I) \subset \{(x,\xi) \in \mathbf{R}^n \times (\mathbf{R}^n\backslash 0) : \xi = \nabla_x\varphi(x,\theta)/|\nabla_x\varphi(x,t)|,\ \nabla_\theta\varphi(x,\theta) = 0 \text{ for some} \theta \ne 0\}$. (The unit direction ξ can be regarded as being in the sphere at infinity.) Hint: truncate by a smooth cutoff function near x and take the Fourier transform to find an sum of oscillatory integral expression of the form of as in in the variables (x,θ) with parameters ξ. See Hörmander, Vol I, Theorem 8.1.9.

Remark: If the function a vanishes or is rapidly decreasing along with its derivatives in a conical neighborhood in θ, then the corresponding point (x,ξ) is not in the wave front set. This is a general instance of the calculate of the wave front set of $u(f)$ in 1(d).