18.155 Problem Set 6

Due Wednesday, 10/29/08

1. a) Fix a complex number z. Note that for $Re\, z > -1$, x_+^z is a locally integrable function on $\mathbf{R}$, and hence defines a homogeneous distribution of degree z. Show that for $z \neq -1, -2, \dots$, the formula

$$x_+^z = \frac{1}{(z+1)(z+2)(z+3)\cdots(z+n)} \frac{d^n}{dx^n} x_+^{z+n}$$

is a consistent definition of a distribution $Re\, z > -1 - n$. (Use integration by parts and analytic continuation to show that different definitions agree.)

b) Show that

$$u_z = \lim_{\epsilon \to 0^+} (x + i\epsilon)^z$$

exists in the sense of distributions, with the complex power defined using the standard branch of the logarithm, $x \in \mathbf{R}$,

$$(x + i\epsilon)^z = e^{z \log|x+i\epsilon| + iz \arg(x+i\epsilon)} \qquad 0 \leq \arg(x + i\epsilon) \leq \pi, \quad \epsilon > 0$$

The distribution u_z is usually denoted $(x + i0)^z$. Similarly, $v_z = (x - i0)^z$ is the limit of $(x - i\epsilon)^z$ as $\epsilon \to 0^+$. Write $u_z(\varphi)$ as a convergent integral for $\varphi \in \mathcal{S}(\mathbf{R})$. Express u_z and v_z as linear combinations of the distributions in part (a) provided $z \neq -1, -2, \dots$. For $z = -1, -2, \dots$, express them as linear combinations of the appropriate number of derivatives of p.v$(1/x)$ and δ.

c) Optional: Show that the space of homogeneous distributions on $\mathbf{R}$ of degree z is two-dimensional. Thus we have given two different bases for the space in parts (a) and (b) above.

d) Using part (c), find the Fourier transforms of the distributions $(x \pm 0i)^z$ in terms of whichever basis you prefer.

e) Consider a homogeneous distribution u in one variable, a smooth function $h : \mathbf{R}^n \to \mathbf{R}$, and a point $p \in \mathbf{R}^n$ at which $h(p) = 0$ and $\nabla h(p) \neq 0$. Make sense out of the distribution v on $\mathbf{R}^n$ that we write informally as $u(h(x))$, $x \in \mathbf{R}^n$, in a sufficiently small neighborhood U of p. Prove that its wave front set satisfies

$$(U \times S^{n-1}) \cap \mathrm{WF}(v) \subset (U \times S^{n-1}) \cap \{(x, \theta) : h(x) = 0, \quad \theta = \pm \nabla h(x)/|\nabla h(x)|\}$$

Explain when the inclusion is an equality and what else is possible.

2. In this problem we describe all homogeneous, rotation-invariant solutions to the equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta \right) F = c\delta(x, t)$$

for some c. These form a three-dimensional family. (See Hörmander vol 1, Section 6.2.)

Consider $(t, x) \in \mathbf{R} \times \mathbf{R}^n$ and $(\tau, \xi) \in \mathbf{R} \times \mathbf{R}^n$. Define the partial Fourier transforms on $\varphi \in \mathcal{S}$ by

$$\mathcal{F}_1\varphi(\tau, x) = \int_{\mathbf{R}} \varphi(t, x) e^{-it\tau} dt; \qquad \mathcal{F}_2\varphi(t, \xi) = \int_{\mathbf{R}^n} \varphi(t, x) e^{-ix\cdot\xi} dx$$

and for $u \in \mathcal{S}'$ by

$$\mathcal{F}_1 u(\varphi) = u(\mathcal{F}_1\varphi) \quad \text{and} \quad \mathcal{F}_2 u(\varphi) = u(\mathcal{F}_2\varphi)$$

a) Define E a distribution on $\mathbf{R} \times \mathbf{R}^n$ by $\mathcal{F}_2 E = H(t)\sin(t|\xi|)/|\xi|$, where H is the Heaviside function. Show that $[(d/dt)^2 + |\xi|^2]\mathcal{F}_2 E = c\delta(t)$ and calculate the constant c. Explain why

$$E(\varphi) = \mathcal{F}_2 E(\mathcal{F}_2^{-1}\varphi)$$

and deduce that

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) E = c\delta(x,t)$$

b) Let G be a solution to the wave equation $((\partial/\partial t)^2 - \Delta)G = 0$ that is rotation invariant in x and a homogeneous distribution of degree $1-n$. Find the ordinary differential equation satisfied by the partial Fourier transforms $\mathcal{F}_2 G(t,\xi)$ and use the homogeneity and rotation invariance to show that $\mathcal{F}_2 G$ is a linear combination of $e^{it|\xi|}/|\xi|$ and $e^{-it|\xi|}/|\xi|$ in the case $n > 1$.

c) Denote by $G_\pm$ the distributions such that

$$\mathcal{F}_2 G_\pm = e^{\pm it|\xi|}/|\xi|$$

Find $\mathcal{F}_1\mathcal{F}_2 G_\pm$, and explain how these are defined as homogeneous distributions for $n > 1$. (Hint: They satisfy $(\tau^2 - |\xi|^2)U = 0$. Use the formula from the last part of Problem 1 for $(\tau, \xi) \neq (0,0)$ and extend to the origin with the help of the homogeneity.)

d) Find the two-dimensional family of solutions corresponding to parts (b) and (c) in the case $n = 1$.

e) Show that when n is even, the three homogeneous distributions

$$(t^2 - |x|^2)_+^{(1-n)/2} H(t), \quad (t^2 - |x|^2)_+^{(1-n)/2} H(-t), \quad (t^2 - |x|^2)_-^{(1-n)/2}$$

satisfy equations of the form

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) F = c\delta(x,t)$$

for some constant c, without evaluating the constant. Similarly, when $n = 2k+1$ is odd, $k \geq 1$, show that the three homogeneous distributions

$$\delta^{(k-1)}(t^2 - |x|^2)H(t), \quad \delta^{(k-1)}(t^2 - |x|^2)H(-t), \quad (t^2 - |x|^2 + i0)^{-k} = \lim_{z \to -k}\left[(t^2 - |x|^2)_+^z + (-1)^k (t^2 - |x|^2)_-^z\right]$$

satisfy the wave equation with right hand side a multiple of the delta function. ($\delta^{(k)}$ denotes the kth derivative of the delta function in one variable and $\delta^{(0)} = \delta$.)

f) Show that for suitable choice of $\pm$ depending on n odd/even,

$$G = (t^2 - |x|^2 + i0)^{(1-n)/2} H(t) \pm (t^2 - |x|^2 - i0)^{(1-n)/2} H(-t), \quad (n > 1)$$

is well-defined homogeneous distribution, smooth across $t = 0$.

g) Show that G is a linear combination of the solutions in part (e) and, by considering the wave front sets, identify the Fourier transform of G up to a constant.

h) Evaluate all the constants and find all Fourier transforms of the functions in parts (e) and (f) using the constant you evaluated in part (a). Hint: E has compact support in x for each fixed t; why do we know this?

3. a) Find the wave front set of the distributions in part (e) above.

b) Optional: Find the scattering wave front set of these solutions.