18.155 Problem Set 5

Due Wednesday, 10/15/08

This problem set is shorter than usual because Monday, Oct 13, is a holiday.

1. a) Show that a distribution in $\mathbf{R}^n$ is homogeneous of degree z if and only if it satisfies

$$(x \cdot \nabla - z)\, u = 0$$

in the sense of distributions.

b) Show that if u is a homogeneous distribution with singular support at the origin, then the same is true of its Fourier transform.

2. a) Formulate and prove analogous statements to Problem 1b for homogeneity with respect to parabolic dilations, $(x, t) \to (rx, r^2 t)$, $r > 0$, $x \in \mathbf{R}^n$, $t \in \mathbf{R}$.

b) Write down the Fourier transform of the fundamental solution to the heat operator $(\partial_t - \Delta)$ that is homogeneous with respect to parabolic dilations. Without computing its inverse Fourier transform, use part (a) to prove that the heat operator is to prove that the operator is hypoelliptic.

c) Now compute the inverse Fourier transform that you avoided computing in part (b). The formula for the homogeneous fundamental solution can be guessed using the initial value problem solution and Duhamel's principle or otherwise. (See Rauch book for a discussion of the Duhamel principle and Melrose exercises 60 and 64.)

3. a) For $(x, t) \in \mathbf{R}^2$, define $E(x, t) = 1$ for $|x| < t$ and $E(x, t) = 0$ otherwise. Show that E is a (multiple of a) fundamental solution of the wave equation: $(\partial_t^2 - \partial_x^2)E = c\delta$ (and find the constant c).

b) Calculate $\mathrm{ss}(E)$, the singular support of E.

c) Calculate the Fourier transform $\hat{E}$. (Recall that the Fourier transform of the Heaviside function is expressed in using either the principle value of $1/x$ or $1/(x \pm i0)$. Ultimately a distribution must be expressed in terms of a convergent integral.)