18.155 Problem Set 4

Due Wednesday, 10/08/08

1. For $\varphi \in \mathcal{D}$, define $T_x\varphi$ and $\tilde{\varphi}$ by $(T_x\varphi)(y) = \varphi(y-x)$ and $\tilde{\varphi}(x) = \varphi(-x)$. Let $u \in \mathcal{D}'$ and define the convolution of u and φ by

$$u * \varphi(x) = u(T_x(\tilde{\varphi})) = u(\varphi(x - \cdot))$$

We remarked in lecture that $u*\varphi(x)$ is infinitely differentiable and $\partial^\alpha(u*\varphi) = u*(\partial^\alpha\varphi)$. It follows that the linear operator $L_u(\varphi) = u*\varphi$ from $\mathcal{D}$ to C^∞ is continuous in the sense that if the supports of φ_j are contained in a fixed compact set and for each α, $\partial^\alpha\varphi_j$ tends to zero uniformly in x, then $\partial^\alpha L_u(\varphi_j)$ tends to zero uniformly on compact sets as $j \to \infty$.

a) Show, conversely, that every linear mapping L from $\mathcal{D}$ to C^∞ that is continuous in this sense and translation invariant

$$T_x(L(\varphi)) = L(T_x(\varphi)) \quad \text{for all } x \in \mathbf{R}^n$$

is represented by a unique distribution, i. e., $L = L_u$ for a unique distribution u.

b) If u_1 and u_2 are distributions and one or the other has compact support, then we define $L(\varphi) = u_1 * (u_2 * \varphi)$. Show that L is a continuous linear operator in the sense above and confirm that it is translation invariant. (It follows from part (a) that there is a unique distribution u such that $L(\varphi) = u * \varphi$ and we define $u_1 * u_2 = u$.)

2. a) Solve the initial value problem $u_t - u_{xxx} = 0$ in $t > 0$ and $u(x,0) = f(x)$ for $f \in \mathcal{S}(\mathbf{R})$ using the Fourier transform and its inverse. Show that u is infinitely differentiable.

b) Write the solution in the form $u(\cdot,t) = K_t * f(x)$, where K_t is given as an (inverse) Fourier transform.

c) Show that K_1 is infinitely differentiable and estimate the growth as $x \to \pm\infty$ of $K_1(x)$ and its derivatives. Do this by writing

$$K_1(x) = \frac{1}{2\pi}\sum_{j=0}^{\infty} F_j(x)$$

where

$$F_j(x) = \int_{-\infty}^{\infty} \psi_j(\xi)\hat{K}_1(\xi)e^{ix\xi}d\xi$$

and the functions ψ_j form what is known as a dyadic partition of unity of ξ-space. In other words,

$$\sum_{j=0}^{\infty}\psi_j(\xi) = 1$$

of smooth functions ψ_j on the real line supported in $2^{j-2} < |\xi| < 2^{j+2}$ for $j = 1, 2, \ldots$ and such that ψ_0 is supported in $|\xi| < 1$, with the properties

$$|(d/d\xi)^k\psi_j(\xi)| \le C_k 2^{-kj}$$

Hint: Use integration by parts to find the best bounds on $F_j(x)$ and its derivatives before adding up. Then choose the bound with the best order of magnitude for each summand F_j. The correct choice will depend both on the relationship between j and x.

d) Show that $K_t(x) = t^{-1/3}K_1(t^{-1/3}x)$ by first proving the analogous statement in the sense of distributions and then using part (c) to see that the equation as written makes sense. Confirm that K_1 satisfies an ordinary differential equation

$$A'' + cxA = 0 \qquad \text{(known as the Airy equation)}$$

for some complex number c. Differentiate this equation k times and compare the bounds you found in part (c) with these equations to be sure they are consistent.

e) Show that for $t > 0$, u satisfies the equation in the sense of distributions for all $f \in L^2(\mathbf{R})$. Show that the boundary values are attained in the sense that

$$\lim_{t\to 0} \|u(\cdot,t) - f\|_{L^2} = 0$$

f) Show that $\displaystyle\int |u(x,t)|^2 dx$ is independent of t.

g) Show that $\displaystyle\int u(x,t)dx$ is independent of t. (Explain for which initial conditions f this expression is finite.)

2. a) Use analytic continuation and the formula for the Fourier transform of the Gaussian (the case $s = 0$ to compute

$$\int_{-\infty}^{\infty} e^{-ixy-(r+is)y^2} dy$$

for all $r > 0$ and all real numbers x and s. Specify which branch of the square root you are using in your formula.

b) For $s > 0$ and $s < 0$, compute

$$\lim_{r\to 0^+} \int_{-\infty}^{\infty} e^{-ixy-(r+is)y^2} dy$$

c) Find a formula for the solution to Schrödinger's equation, $i\partial_t u + \Delta u = 0$ with initial initial condition $u(x,0) = g(x)$, $g \in \mathcal{S}$, the Schwartz class of the form

$$u(x,t) = g * K_t(x) = \int_{\mathbf{R}^n} g(x')K_t(x-x')dx'$$

To obtain the formula, use the Fourier transform to solve $(a+i)\partial_t u + \Delta u = 0$ with $a > 0$, then take the limit as $a \to 0^+$. After finding a formula for K_t, prove that the solution u does satisfy the pde and that $u(x,t) \to g(x)$ as $t \to 0$ both pointwise and in $L^2(\mathbf{R}^n)$ norm.

d) Prove that

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^n)} \le Ct^{-n/2}\|g\|_{L^1(\mathbf{R}^n)}$$

and find the best constant C.

e) Denote $R_t g(x) = u(x,t) = g * K_t(x)$. Show that for all $t \in \mathbf{R}$

$$\|R_t g\|_{L^2} = \|g\|_{L^2}$$

so that R_t extends to a bounded operator on $L^2(\mathbf{R}^n)$. Show that $R_{t_1} \circ R_{t_2} = R_{t_1+t_2}$.

f) Show that for $g \in \mathcal{S}$,

$$\lim_{t\to\infty} \int_{\mathbf{R}^n} \left| u(x,t) - c_n e^{-|x|^2/4it} t^{-n/2} \hat{g}(x/2t) \right|^2 dx = 0$$

(Compute the dimensional constant c_n.) Deduce that for any cone Γ in $\mathbf{R}^n$,

$$\lim_{t\to\infty} \int_\Gamma |u(x,t)|^2 dx = 2^n c_n^2 \int_\Gamma |\hat{g}(\xi)|^2 d\xi$$

(One can also prove these properties for $g \in L^2$. See page 148 of J. Rauch's text.)