18.155 Problem Set 3

Due Wednesday, 10/01/08

1. a) Write formula for the solution to the initial value problem $u_t - \Delta u = 0$ in $t > 0$ and $u(x, 0) = f(x)$ for $f \in \mathcal{S}$ using the Fourier transform and its inverse.

b) The solution you found in part (a) makes sense for initial values f in $H^s(\mathbf{R}^n)$ (for each fixed s, $-\infty < s < \infty$). Use the Sobolev embedding theorem to show that this solution is infinitely differentiable for $t > 0$ and satisfies the heat equation. Moreover,

$$\lim_{t \to 0} \|u(\cdot, t) - f\|_{H^s} = 0$$

(We will discuss uniqueness later.)

c) Write the solution in the form $u(\cdot, t) = K_t * f(x)$, and show that for $t > 0$, u satisfies the heat equation and is infinitely differentiable for all $f \in L^1(\mathbf{R}^n)$. Show that the boundary values are attained in the sense that

$$\lim_{t \to 0} \|u(\cdot, t) - f\|_{L^1} = 0$$

2. Truncation of the Sobolev spaces of first derivatives. Truncation is used especially in connection with first and second order nonlinear partial differential equations.

Consider the Sobolev space $W^{1,p} = W^{1,p}(\mathbf{R}^n)$ of real-valued functions $u \in L^p(\mathbf{R}^n)$ such that $\partial_j u \in L^p(\mathbf{R}^n)$ in the sense of distributions.

a) Define

$$F_\epsilon(t) = (t^2 + \epsilon^2)^{1/2} - \epsilon, \quad t > 0; \quad F_\epsilon(t) = 0, \quad t \le 0$$

Let $u \in W^{1,p}$, $1 \le p \le \infty$. Define $u^+(x) = \max(u(x), 0)$. Show that $\lim_{\epsilon \to 0} F_\epsilon(u) = u^+$ in the sense of distributions.

b) Deduce that if $u \in W^{1,p}$, then $u^\pm \in W^{1,p}$ and

$$\partial_j u^+ = \chi_{\{u>0\}} \partial_j u; \quad \partial_j u^- = -\chi_{\{u<0\}} \partial_j u$$

where $u^-(x) = \max(-u(x), 0)$. Moreover,

$$\partial_j u = 0 \quad \text{almost everywhere on } \{x : u(x) = 0\}$$

and

$$\partial_j |u| = \operatorname{sgn}(u) \partial_j u;$$

c) Define $T_s(f)$ as the inverse Fourier transform of $(1 + |\xi|^2)^{s/2} \hat{f}(\xi)$. and $L^p_s = \{f \in \mathcal{S}' : T_s f \in L^p(\mathbf{R}^n)$. Show that $H^1 = L^2_1$ consists of complex combinations of (the real-valued) functions in $W^{1,2}$. (The analogous statement is true for L^p, $1 < p < \infty$, and follows from what is known as the theory of singular integrals. But it is false for $p = 1$ and $p = \infty$.)

3. a) Show that there is a function in $H^{1/2}(\mathbf{R})$ that is unbounded (say at $x = 0$).

b) Show that there are functions f and g in $H^{1/2}$ such that their product fg does not belong to $H^{1/2}$. (In what sense is their product well defined?)

c) Optional (no due date): Show that the product of functions in $H^{1/2}(\mathbf{R}) \cap L^\infty(\mathbf{R})$ is in $H^{1/2}(\mathbf{R}) \cap L^\infty(\mathbf{R})$. (This is subtle and will be discussed later.)