18.155 Problem Set 2

Due Wednesday, 9/24/08

You have one more chance to turn in the problems from PS1. In particular, here are some further hints for Problem 6 on PS1, restated slightly differently here as Problem 2.

i) To find the Fourier transform of the signum function $\mathrm{sgn}(x)$ note that its derivative is 2δ and take the Fourier transform. This identifies the Fourier transform up to an additive multiple of the delta function (why?) To identify the last additive constant, note that the signum is an odd function. (Symmetries can help a lot. Denote $\tilde{\phi}(x) = \phi(-x)$. We say a distribution u is odd if $u(\tilde{\phi}) = -u(\phi)$ for all test functions $\phi \in \mathcal{D}$ — even if $u(\tilde{\phi}) = u(\phi)$.)

ii) To get the form of most of the rest of the functions, you can consider differential equations they satisfy, up to certain constants. In particular, one can describe (up to constants) the Fourier transform of homogeneous distributions using the differential equations or directly from the definition below in Problem 1. Homogeneity is another form of symmetry, symmetry under dilation.

iii) To evaluate the constants, take odd and even parts and evaluate the distributions on e^{-tx^2}. See remarks in lecture.

1. A distribution u on $\mathbf{R}^n$ is called *homogeneous of degree* z if

$$u(\phi_t) = t^z u(\phi)$$

for all $t > 0$ and all $\phi \in \mathcal{D}$, where $\phi_t(x) = t^{-n}\phi(x/t)$

a) Show that homogeneous distributions are tempered distributions.

b) Show that $\hat{\phi}_t(\xi) = \hat{\phi}(t\xi)$ and that the Fourier transform of a homogeneous distribution of degree z on $\mathbf{R}^n$ is a homogeneous distribution of degree $-n - z$.

c) Show that u is a homogeneous distribution on $\mathbf{R}$ if and only if

$$xu' - zu = 0$$

in the sense of distributions.

d) Identify all homogeneous distributions of degree -1 on $\mathbf{R}$.

2. Compute the Fourier transforms of these tempered distributions on $\mathbf{R}$ and find the relationships among them and their Fourier transforms.

a) δ, the Dirac delta function at the origin.

b) $\mathrm{sgn}(x)$ and $H(x)$ (signum function and Heaviside function).

c) the principal value, p. v. $1/x$

d) $\log|x|$.

e) $1/(x \pm 0i) = \lim_{t\to 0^+} 1/(x \pm it)$

f) $T_a(\phi) = \displaystyle\int_{-a}^{a} \frac{\phi(x) - \phi(0)}{|x|}dx + \int_{|x|>a} \frac{\phi(x)}{|x|}dx$ (Hint: confirm that $T_1 = (d/dx)[\mathrm{sgn}(x)\log|x|]$.)

g) $\lim_{a\to -1}(x_\pm^a - c_a\delta)$, $a > -1$

(Find c_a so that this distributional limit is well-defined. Notation: $x_+ = x$ for $x \geq 0$ and $x_+ = 0$ for $x < 0$; $x_- = -x$ for $x \leq 0$ and $x_- = 0$ for $x > 0$.)

3. a) Solve for the Fourier transform $\hat{u}$ in the x-variable of Laplace's equation in $\mathbf{R}^n \times \mathbf{R}_+$,

$$(\partial_y^2 + \Delta)u(x,y) = 0$$

with boundary value $u(x,0) = g(x)$ ($g \in \mathcal{S}(\mathbf{R}^n)$), in the form $\hat{u}(\xi, y) = M_y(\xi)\hat{g}(\xi)$. The solution need not be unique. Which one should you choose and why?

b) Compute $P_y(x)$ satisfying $\hat{P}_y(\xi) = M_y(\xi)$ as follows.

i) Evaluate the contour integral

$$e^{-|s|} = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{isx}\frac{1}{x^2+1}dx$$

ii) Subsitute $\dfrac{1}{x^2+1} = \displaystyle\int_0^\infty e^{-(1+x^2)u}du$ to confirm that

$$\sqrt{\pi}e^{-|s|} = \int_0^\infty e^{-u}e^{-s^2/4u}u^{-1/2}du$$

iii) Find $h_y(t)$ such that

$$e^{-y|\xi|} = \int_0^\infty h_y(t)e^{-t|\xi|^2}dt$$

iv) Use this superposition of Gaussians to compute the inverse Fourier transform of $M_y(\xi)$.

c) Show that for $g \in C_0(\mathbf{R}^n)$, $u(x,y) = P_y * g(x)$ solves the Laplace equation in $y > 0$ and $u(x,y) \to g(x)$ as $y \to 0$. The function P_y is known as the Poisson kernel.